THE

MATHEMATICAL GAZETTE.

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NOTICE.

THE MATHEMATICAL ASSOCIATION TEACHING COMMITTEE wishes to draw the attention of examiners to the following points about GEOMETRY EXAMINATIONS:

Many examinations already follow the practices suggested.

I. When a CONSTRUCTION is asked for, it should be clearly stated, either in the question or as a general heading,

(i) What instruments may be used.
(ii) Whether the construction is to be fully described in words, or only those parts of it which are not obvious from the figure.
(iii) Whether a theoretical proof is required.

II. When the proof of a THEOREM is asked for, if the examiner wishes for an accurate figure drawn with instruments, this should be stated in the question.

III. The Committee is of opinion that, in general, it is inadvisable to require that the figures for the proofs of theorems should be drawn accurately with instruments; the Committee considers that such a requirement tends to waste of time, and that a neatly drawn freehand figure should suffice.—Signed on behalf of the Committee,

E. W. HOBSON,
April, 1911. *President of the Mathematical Association.*

[The above has been sent out to Examining Bodies by the Secretary of the Committee.]

ON TERTIAL, QUINTAL, ETC., FRACTIONS.

BY LT.-COL. ALLAN CUNNINGHAM, R.E., FELL. OF KING'S COLL. LOND.

1. DEF. The reciprocal $(1/N)$ of any number (N) will be termed a *Binal*,* *Tertial*, *Quintal*, ... *Decimal*, ... *r-mal* fraction, when expressed in

*The subject of Binal Fractions was treated of in the author's previous paper on *Binal Fractions* (*Math. Gazette*, vol. iv., 1908, pp. 259-267). For facility of comparison, the present paper is written in the *same style* as the former, and the corresponding Articles and Results bear the same reference numbers. [The author's acknowledgments are due to Mr. M. J. Woodall, A.R.C.Sc., for help in reading the proof sheets.]

the scale whose radix is $r=2, 3, 4, \ldots 10, \ldots r$ respectively. In this paper the developments will be given for the most part in a general form in the r-scale; the examples will be in the scales of $r=3$ and 5. The notation is similar to that of ordinary *decimals*. Thus (see Tables at end):

1°. A "point" (the tertial, quintal, etc., *point*) marks the beginning of the (tertial, quintal, etc.) fraction.

2°. The tertial, quintal, etc. $(1/N)$ consists of a certain sequence of ciphers and other digits (<3, 5, etc., r) to the right of the "point"; partly non-recurrent, and partly *recurrent* (all recurrent in the Tables at end).

3°. The beginning and end of the recurrent portion (or *repetend*) are marked by a point placed over the first and last figure of the repetend.

2. *N a power of r.* Let $N=r^a$; then

$$1/N = \cdot 000 \ldots 01, \text{ [}(a-1) \text{ ciphers, one unit]}, \quad \ldots\ldots(1)$$

and, note that there is *no recurrence*. $\ldots\ldots(1a)$

Simple non-repetends. A number of these (non-repetend) forms may be written down at sight.

$$\frac{1}{r} \cdot \frac{r^a-1}{r-1} = \cdot 111 \ldots 1, \text{ [}a \text{ units]}. \quad \ldots\ldots(1b)$$

$$\frac{1}{r^{2a}} \cdot \frac{r^{2a}-1}{r^2-1} = \cdot 0101 \ldots 01, \text{ [}a \text{ repetitions]}. \quad \ldots\ldots(1c)$$

$$\frac{1}{r^{3a}} \cdot \frac{r^{3a}-1}{r-1} = \cdot 001001 \ldots 001, \text{ [}a \text{ repetitions]}, \quad \ldots\ldots(1d)$$

and so on.

Also, writing $q=(r-1)$, $q_2=(r-2)$, $q_3=(r-3)$, etc., as *digits*: $\ldots\ldots(1e)$

$$\frac{1}{r}(r-1) = \cdot q, \quad \frac{1}{r^2} \cdot (r^2-1) = \cdot qq. \quad \ldots\ldots(1f)$$

$$\frac{1}{r^a} \cdot (r^a-1) = \cdot qqq \ldots q, \text{ [}a \text{ digits]}. \quad \ldots\ldots(1g)$$

$$\frac{1}{r^q} \cdot \frac{r^q-1}{q^q} = \cdot 0123 \ldots q_4 q_3 q, \text{ [}q \text{ digits, no } q_2\text{]}. \quad \ldots\ldots(1h)$$

Also,

$$\frac{1}{r^3} \cdot \frac{(r^3+1)}{r+1} = \cdot q1, \quad \frac{1}{r^4} \cdot \frac{q^5+1}{r+1} = \cdot q0q1. \quad \ldots\ldots(1i)$$

$$\frac{1}{r^{2a}} \cdot \frac{r^{2a+1}+1}{r+1} = \cdot q0q0 \ldots q1, \text{ [}a \text{ digits}=q\text{]}. \quad \ldots\ldots(1j)$$

2a. *Pure repetends.* Let $N=A$, an *integer* prime to r.

$1/N$ consists *solely of a repeating cycle*. $\ldots\ldots(2)$

2b. *Multiples of powers of radix.* Let $N=r^a \cdot A$, where A is an integer prime to r, and let P be the period of $1/A$ which is *wholly repetend*; then

$1/N$ consists of a group of a ciphers following the r-mal "point" non-recurrent, due to the factor r^a, followed by the period P of $1/A$, which is the repetend. $\ldots(3)$

Hence, it is unnecessary to consider multiples of r^a any further, as their reciprocals can be at once written down by the above rule, when the repetend cycle of $1/A$ is known.

3. *Notation.*

Let N be an integer prime to r, and $>r^a$, but $<r^{a+1}$.

„ P be the repetend cycle of $1/N$ in the r-scale.

„ n = the number of digits in P.

„ c = number of leading ciphers (*i.e.* following the "point") in P.

Let v = number of digits in P following the leading ciphers.

„ ξ be the Haupt-Exponent* of r to mod N, *i.e.* the *least* exponent giving $r^{\xi} \equiv +1 \pmod{N}$.

„ $q = r - 1$.

3a. *Number of figures* (n) *in period* (l'). By the above,

$$c + v = n = \xi. \quad \ldots\ldots (4)$$

$$c = a. \quad \ldots\ldots (5)$$

Thus, in all cases, the leading a figures (next to the "point") are ciphers, followed by a digit >0; there are $(\xi - a)$ digits following those leading ciphers, and none of them can exceed $q = (r-1)$.

4. *Fractional form of* $1/N$. By definition, P is one period of the repetend of $1/N$, expressed as an integer in the r-scale, and contains ξ figures. Hence $P \div r^{\xi}$ is the real value of the first period of $1/N$, and therefore

$$\frac{1}{N} = \frac{P}{r^{\xi}} + \frac{P}{r^{2\xi}} + \frac{P}{r^{3\xi}} + \text{etc. } ad.\ inf. \quad \ldots\ldots (7)$$

$$\therefore\ r^{\xi} \cdot \frac{1}{N} = P + \left(\frac{P}{r^{\xi}} + \frac{P}{r^{2\xi}} + \ldots\ ad.\ inf. \right) = P + \frac{1}{N}.$$

$$\therefore\ P = \frac{r^{\xi} - 1}{N}. \quad \ldots\ldots (8)$$

$$\therefore\ P \text{ is the quotient of } (r^{\xi} - 1) \div N \text{ in the } r\text{-scale}. \quad \ldots\ldots (8a)$$

The quantity (P), thus found, will be an integer, containing only $\alpha = (\xi - a)$ digits (the first and last of which are >0). A group of a ciphers with the "points" on the left should be prefixed to form the r-mal fraction P/r^{ξ}, which is one period of $1/N$. If a point be now placed over the first and last figures, to show that the whole group is recurrent, the result will be the complete or real fractional form of $1/N$.

4a. *Periods of co-factors of* $(r^{\xi} - 1)$. Let N_1, N_2 be a pair of co-factors of $(r^{\xi} - 1)$, and let P_1, P_2 be the periods of $1/N_1$, $1/N_2$ respectively; then

$$P_1 = \frac{r^{\xi} - 1}{N_1} = N_2; \text{ and } P_2 = \frac{r^{\xi} - 1}{N_2} = N_1, \text{ [all in } r\text{-scale]}. \quad \ldots\ldots (9)$$

Ex.† Scale of $r = 3$. Take $n = 6$; then $3^6 - 1 = 26 \,.\, 28 = N_1 \,.\, N_2$.

P_1 of $26 = \frac{1}{13}(3^6 - 1) = 28 = (1001$ in 3-scale),

P_2 of $28 = \frac{1}{7}(3^6 - 1) = 26 = (222$ in 3-scale).

$\therefore\ \frac{1}{13} = \cdot\dot{0}0100\dot{1} = \ddagger \cdot \dot{0}0\dot{1}$, $\frac{1}{7} = \cdot\dot{0}0022\dot{2}$ (in 3-scale).

Ex.† Scale of 5. Take $n = 5$; then $5^5 - 1 = 44 \,.\, 71 = N_1 \,.\, N_2$.

P_1 of $44 = \frac{1}{4}(5^5 - 1) = 71 = (241$ in 5-scale),

P_2 of $71 = \frac{1}{71}(5^5 - 1) = 44 = (134$ in 5-scale).

$\therefore\ \frac{1}{44} = \dot{0}024\dot{1}$, $\frac{1}{71} = \cdot\dot{0}013\dot{4}$ (in 5-scale).

Hence, to find the *least* number (N_n) such that the period (P_2) of $1/N_2$ shall be a *given* number (N_1) in the r-scale, preceded only by ciphers; it suffices to find the Haupt-Exponent (ξ) of r to mod. N_1: thus, by (9), taking

$$N_2 = \frac{1}{N_1} \cdot (r^{\xi} - 1) \text{ gives } P_2 = 1/N_1 \text{ (as required)}. \quad \ldots\ldots (9a)$$

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* In what follows ξ is always supposed *known*. For prime, and powers of prime, moduli ($N = p$ or p^n) < 1000, it can be easily found from Jacobi's *Canon Arithmeticus*.

† Compare Tables at end.

‡ The reduction of $\cdot\dot{0}0100\dot{1}$ to $\dot{0}0\dot{1}$ is due to the fact that $N_1 = (3^3 - 1)$, for which $\xi = 3$.

Although the Rule (9) just given suffices for computing the complete period (P) of the r-mal fraction of the reciprocal ($1/N$) of any integer (N), it is thought that an investigation of the properties of these fractions, with examples for the radices $r=3, 5$, will be interesting: this will be found to give in many cases simpler methods for finding $1/N$.

5. *Simple Cases* $[N=(r^a \mp 1)]$. The periods (P) can here be written down *at sight.*

$$N=r^a-1; \quad 1/N=\dot{0}00\ldots0\dot{1}, \qquad [(a-1) \text{ ciphers, 1 unit}]. \ldots\ldots(10a)$$

$$N=r^a+1; \quad 1/N=\dot{0}00\ldots00qq\ldots\dot{q}, \quad [a \text{ ciphers, } a \text{ digits}=q]. \ldots\ldots(10b)$$

$$N=r^{a+1}-1; \quad 1/N=\dot{0}00\ldots00\dot{1}, \qquad [a \text{ ciphers, 1 unit}]. \ldots\ldots(10c)$$

Hence the reciprocal ($1/N$) of all numbers (N) from $N=(r^a+1)$ to $N=(r^{a+1}-1)$ have their periods (P) preceded by a group of a ciphers, whilst P itself gradually decreases from $P=qqq\ldots q$ (a digits) when $N=(r^a+1)$ to $P=1$ when $N=(r^{a+1}-1)$.

Another simple form is

$$N=q^2, \text{ then } 1/N=\dot{0}123\ldots q_4q_3\dot{q}, \quad [q \text{ digits, no } q_2]. \ldots\ldots(10d)$$

Compare with this the non-repetend ($1h$) with *same period* (P).

6. *Factors of* $(r^{2a}-1)$. Let $R=r^a$, and let N_x be defined by the succession-formula

$$N_{x+1}=R.N_x+1=r^a.N_x+1, \quad [\text{with } N_0=1]. \ldots\ldots(11)$$

Then
$$N_x=R^x+R^{x-1}+\ldots+R^2+R+1 \ldots\ldots(11a)$$
$$=111\ldots11, \quad [(x+1) \text{ units in } R\text{-scale}]$$
$$=100\ldots0100\ldots0100\ldots0100\ldots01, \quad [\text{in } r\text{-scale}; \ldots\ldots(13)$$
$$(x+1)a \text{ digits}, (x+1) \text{ units}; (a-1) \text{ ciphers in sub-period}]$$
$$=\frac{R^{x+1}-1}{R-1}=\frac{r^{(x+1)a}-1}{r^a-1}. \ldots\ldots(12)$$

Hence N_x, (r^a-1) are co-factors of $(r^{(x+1)a}-1)$.

Hence P_x (of $1/N_x$) $=(r^a-1)=qqq\ldots q$, [a digits in r-scale]. ……(14)

And $1/N_x=\dot{0}00\ldots0qq\ldots\dot{q}$, [$xa$ ciphers, a digits $=q$]. ……(14a)

A few of these numbers, arising in the scales of $r=3, 5$, are shown below with their equivalents in the denary scale.

a	N (denary). $r=3$, $R=3^a$. R	$x=0,\ 1,\ 2,\ 3$	N (denary). $r=5$, $R=5^a$. R	$x=0,\ 1,\ 2,\ 3$	N (in r-scale). [$(x+1)$ units, $(a-1)$ ciphers.]	Period of $1/N$. [xa ciphers, a of q-digits.]
1	3	1, 4, 13, 40	5	1, 6, 31, 156	111 1	$\dot{0}00\ldots0\dot{q}$
2	9	1, 10, 91, 820	25	1, 26, 656, —	10101 101	$\dot{0}00\ldots0q\dot{q}$
3	27	1, 28, 757, —	125	1, 126, — —	1001001 ..1001	$\dot{0}00\ldots0qq\dot{q}$
4	81	1, 82, — —	625	1, 626, — —	100010001 ... 1	$\dot{0}00\ldots0qqq\dot{q}$

It will be seen that the top line has $N=(r^{x+1}-1)\div(r-1)$, and the 2nd column has $N=(r^a+1)$.

6a. $N=2(r^a \mp 1)$, [r *odd*]. The periods can in these cases also be written down at sight.

Let $N'=(r^a-1), \quad N''=(r^a+1).$

Then, writing $b=\frac{1}{2}(r-1), \quad a=\frac{1}{2}(r+1),$

$$1/2N'=\dot{0}00\ldots0bb\ldots b\dot{a}, \quad [a \text{ ciphers}, (a-1) \text{ digits}=b]. \ldots\ldots(14b)$$

$$1/2N''=\dot{0}00\ldots0bb\ldots b\dot{b}, \quad [a \text{ ciphers}, a \text{ digits}=b]. \ldots\ldots(14c)$$

6b. $N=\frac{1}{2^\kappa}\cdot(r^\alpha\mp1)$, [$r$ *odd*]. The periods can in these cases also be written down at sight. [N', N'' as in Art. 6*a*.]

$$1\Big/\frac{1}{2}N'=\cdot\dot0\dot0\ldots0\dot2, \quad [(\alpha-1)\text{ ciphers, one }2;\ r>2]. \quad\ldots\ldots(14d)$$

$$1\Big/\frac{1}{4}N'=\cdot\dot0\dot0\ldots0\dot4, \quad [(\alpha-1)\text{ ciphers, one }4;\ r>4]. \quad\ldots\ldots(14e)$$

$$\vdots$$

$$1\Big/\frac{1}{2^\kappa}\cdot N'=\cdot\dot0\dot0\ldots0\dot k, \quad [(\alpha-1)\text{ ciphers, one }k=2^\kappa,\ r>k]. \quad\ldots\ldots(14f)$$

$$1\Big/\frac{1}{2^\kappa}\cdot N'=\cdot\dot0\dot0\ldots0\,[2^\kappa], \quad [\alpha\text{ figures in }P,\ [2^\kappa]\text{ means }2^\kappa\text{ expressed in }r\text{-scale}]. \quad(14g)$$

$$1\Big/\frac{1}{2}N''=\cdot\dot0\dot0\ldots01qqq\ldots q\dot q_2, \quad [(\alpha-1)\text{ ciphers, }(\alpha-1)\text{ digits}=q]. \quad\ldots\ldots(14h)$$

Ex. Here follow examples arising in the scales of $r=3, 5$; the equivalents of $2N'$, $2N''$; $\frac{1}{2^\kappa}\cdot N'$, $\frac{1}{2^\kappa}N''$ are given in the denary scale.

	Integers (denary).								Reciprocals [in r-scale.]	Reference.
	$r=3$.				$r=5$.					
$\alpha=$	1,	2,	3,	4	1,	2,	3,	4		
$N'=$	2,	8,	26,	80	4,	24,	124,	624	$\cdot\dot0\dot0\ldots0\dot1$	See (10*a*)
$2N'=$	4,	16,	52,	160	8,	48,	248,	—	$\cdot\dot0\dot0\ldots0\dot0bbb\ldots b\dot a$	See (14*b*)
$\frac{1}{2}N'=$	1,	4,	13,	40	2,	12,	62,	312	$\cdot\dot0\dot0\ldots0\dot2$	See (14*d*)
$\frac{1}{4}N'=$	.,	2,	.,	.	1,	6,	31,	78	$\cdot\dot0\dot0\ldots0\dot4$, [$r>4$]	See (14*e*)
$\frac{1}{4}N'=$	.,	2,	.,	20	.,	.,	.,	—	$\cdot\dot0\dot0\ldots0[\dot4]$	See (14*g*)
$\frac{1}{8}N'=$	.,	1,	.,	10	.,	3,	.,	39	$\cdot\dot0\dot0\ldots0[\dot8]$	See (14*g*)
$N''=$	4,	10,	28,	82	6,	26,	126,	626	$\cdot\dot0\dot0\ldots00qqq\ldots q\dot q$	See (10*b*)
$2N''=$	8,	20,	56,	164	12,	52,	252,	—	$\cdot\dot0\dot0\ldots00\dot0bb\ldots b\dot b$	See (14*c*)
$\frac{1}{2}N''=$	2,	5,	14,	41	3,	13,	63,	313	$\cdot\dot0\dot0\ldots qqqq\ldots q\dot q_2$	See (14*h*)

7. *End Figures of P.*

Let $N=r^\alpha+B$, where $B<r^\alpha$.

Let P, Q be the periods of $1/N$ and $1/B$.

Then, $P\equiv Q\pmod{r^\alpha}$, ...(15)

so that

The end group of α figures of $1/N$, $(N>r^\alpha)$, is the *same* as the end group of α figures of $1/B=1/(N-r^\alpha)$. .....(15*a*)

Hence 2α figures of the period (P) of $1/N$ are known for all numbers $N>r^\alpha$, viz. α ciphers at the beginning (Art. 3*a*), and the end group of α figures as above (if that group be known for the period of $1/B$); this leaves only $(\xi-2\alpha)$ figures undetermined. Hence also the *complete period* of $1/N$ may be written down at sight for all numbers $N>r^\alpha$ but $<r^{\alpha+1}$, when $\xi\nrightarrow2\alpha$, if the end group of α figures of the period of $1/B$ is known.

The Table following gives the mod 3-, or 4-figure groups of the periods of $1/N$ for all numbers (N) as follows:

4-figure endings for $r=3$; 3-figure endings for $r=5$.

This Table enables the *complete periods* of $1/N$ to be written down as follows:

In 3-scale, $N\nrightarrow122$, $\xi\nrightarrow8$; in 5-scale, $N\nrightarrow185$, $\xi\nrightarrow6$,

and part of the periods when ξ exceeds 8 or 6 respectively.

(*To be continued.*)

Although the Rule (9) just given suffices for computing the complete period (P) of the r-mal fraction of the reciprocal ($1/N$) of any integer (N), it is thought that an investigation of the properties of these fractions, with examples for the radices $r=3, 5$, will be interesting: this will be found to give in many cases simpler methods for finding $1/N$.

5. *Simple Cases* $[N=(r^a \mp 1)]$. The periods (P) can here be written down *at sight.*

$$N=r^a-1;\quad 1/N=\cdot\dot{0}00\ldots0\dot{1}, \qquad [(a-1)\text{ ciphers, 1 unit}]. \ldots\ldots(10a)$$
$$N=r^a+1;\quad 1/N=\cdot\dot{0}00\ldots00qq\ldots\dot{q}, \quad [a\text{ ciphers, }a\text{ digits}=q]. \ldots\ldots(10b)$$
$$N=r^{a+1}-1;\quad 1/N=\cdot\dot{0}00\ldots00\dot{1}, \qquad [a\text{ ciphers, 1 unit}]. \ldots\ldots(10c)$$

Hence the reciprocal ($1/N$) of all numbers (N) from $N=(r^a+1)$ to $N=(r^{a+1}-1)$ have their periods (P) preceded by a group of a ciphers, whilst P itself gradually decreases from $P=qqq\ldots q$ (a digits) when $N=(r^a+1)$ to $P=1$ when $N=(r^{a+1}-1)$.

Another simple form is

$$N=q^2, \text{ then } 1/N=\dot{0}123\ldots q_4q_3\dot{q}, \quad [q\text{ digits, no }q_2]. \ldots\ldots(10d)$$

Compare with this the non-repetend ($1h$) with *same period* (P).

6. *Factors of* $(r^{xa}-1)$. Let $R=r^a$, and let N_x be defined by the succession-formula

$$N_{x+1}=R.N_x+1=r^a.N_x+1, \text{ [with } N_0=1]. \ldots\ldots(11)$$

Then
$$N_x=R^x+R^{x-1}+\ldots+R^2+R+1 \ldots\ldots(11a)$$
$$=111\ldots11, \text{ [}(x+1)\text{ units in }R\text{-scale]}$$
$$=100\ldots0100\ldots0100\ldots0100\ldots01, \text{ [in } r\text{-scale; } \ldots\ldots(13)$$
$$(x+1)a\text{ digits, }(x+1)\text{ units; }(a-1)\text{ ciphers in sub-period]}$$
$$=\frac{R^{x+1}-1}{R-1}=\frac{r^{(x+1)a}-1}{r^a-1}. \ldots\ldots(12)$$

Hence N_x, (r^a-1) are co-factors of $(r^{(x+1)a}-1)$.

Hence P_x (of $1/N_x$) $=(r^a-1)=qqq\ldots q$, [a digits in r-scale]. ……(14)

And $1/N_x=\cdot\dot{0}00\ldots0qq\ldots\dot{q}$, [$xa$ ciphers, a digits $=q$]. ……(14a)

A few of these numbers, arising in the scales of $r=3, 5$, are shown below with their equivalents in the denary scale.

a	N (denary). $r=3,\ R=3^a$. R	$x=0,\ 1,\ 2,\ 3$	$r=5,\ R=5^a$. R	$x=0,\ 1,\ 2,\ 3$	N (in r-scale). [$(x+1)$ units, $(a-1)$ ciphers.]	Period of $1/N$. [xa ciphers, a of q-digits.]
1	3	1, 4, 13, 40	5	1, 6, 31, 156	111 1	$\cdot\dot{0}00\ldots0\dot{q}$
2	9	1, 10, 91, 820	25	1, 26, 656, —	10101 101	$\cdot\dot{0}00\ldots0q\dot{q}$
3	27	1, 28, 757, —	125	1, 126, — —	1001001..1001	$\cdot\dot{0}00\ldots0qq\dot{q}$
4	81	1, 82, — —	625	1, 626, — —	100010001 ... 1	$\cdot\dot{0}00\ldots0qqq\dot{q}$

It will be seen that the top line has $N=(r^{x+1}-1)\div(r-1)$, and the 2nd column has $N=(r^a+1)$.

6a. $N=2(r^a \mp 1)$, [r *odd*]. The periods can in these cases also be written down at sight.

Let $N'=(r^a-1),\ N''=(r^a+1)$.

Then, writing $b=\frac{1}{2}(r-1),\quad a=\frac{1}{2}(r+1)$,

$$1/2N'=\cdot\dot{0}00\ldots0bb\ldots b\dot{a}, \quad [a\text{ ciphers, }(a-1)\text{ digits}=b]. \ldots\ldots(14b)$$
$$1/2N''=\cdot\dot{0}00\ldots0bb\ldots b\dot{b}, \quad [a\text{ ciphers, }a\text{ digits}=b]. \ldots\ldots(14c)$$

6b. $N=\frac{1}{2^k}\cdot(r^a\mp1)$, [$r$ *odd*]. The periods can in these cases also be written down at sight. [N', N'' as in Art. 6*a*.]

$$1\Big/\frac{1}{2}N'=\cdot\dot{0}00\ldots00\dot{2}, \quad [(a-1)\text{ ciphers, one }2;\ r>2]. \qquad \ldots\ldots(14d)$$

$$1\Big/\frac{1}{4}N'=\cdot\dot{0}00\ldots00\dot{4}, \quad [(a-1)\text{ ciphers, one }4;\ r>4]. \qquad \ldots\ldots(14e)$$

$$\vdots$$

$$1\Big/\frac{1}{2^k}\cdot N'=\cdot\dot{0}00\ldots00\dot{k}, \quad [(a-1)\text{ ciphers, one }k=2^\kappa,\ r>k]. \qquad \ldots\ldots(14f)$$

$$1\Big/\frac{1}{2^\kappa}\cdot N'=\cdot\dot{0}00\ldots0\,[2^\kappa], \quad [a\text{ figures in }P,\ [2^\kappa]\text{ means }2^\kappa\text{ expressed in }r\text{-scale}]. \ (14g)$$

$$1\Big/\frac{1}{2}N''=\cdot\dot{0}00\ldots01qqq\ldots q\dot{q}_2, \quad [(a-1)\text{ ciphers, }(a-1)\text{ digits}=q]. \qquad \ldots\ldots(14h)$$

Ex. Here follow examples arising in the scales of $r=3$, 5; the equivalents of $2N'$, $2N''$; $\frac{1}{2^\kappa}\cdot N'$, $\frac{1}{2^\kappa}N''$ are given in the denary scale.

	Integers (denary).								Reciprocals [in r-scale.]	Reference.
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$a=$	1,	2,	3,	4	1,	2,	3,	4		
$N'=$	2,	8,	26,	80	4,	24,	124,	624	$\cdot\dot{0}00\ldots0\dot{1}$	See (10*a*)
$2N'=$	4,	16,	52,	160	8,	48,	248,	—	$\cdot\dot{0}00\ldots00bbb\ldots b\dot{a}$	See (14*b*)
$\frac{1}{2}N'=$	1,	4,	13,	40	2,	12,	62,	312	$\cdot\dot{0}00\ldots0\dot{2}$	See (14*d*)
$\frac{1}{4}N'=$	.,	2,	.,	.	1,	6,	31,	78	$\cdot\dot{0}00\ldots0\dot{4}$ $[r>4]$	See (14*c*)
$\frac{1}{4}N'=$	.,	2,	.,	20	.,	.,	.,	—	$\cdot\dot{0}00\ldots0[\dot{4}]$	See (14*g*)
$\frac{1}{8}N'=$	.,	1,	.,	10	.,	3,	.,	39	$\cdot\dot{0}00\ldots0[\dot{8}]$	See (14*g*)
$N''=$	4,	10,	28,	82	6,	26,	126,	626	$\cdot\dot{0}00\ldots00qqq\ldots q\dot{q}$	See (10*b*)
$2N''=$	8,	20,	56,	164	12,	52,	252,	—	$\cdot\dot{0}00\ldots00bbb\ldots b\dot{b}$	See (14*c*)
$\frac{1}{2}N''=$	2,	5,	14,	41	3,	13,	63,	313	$\cdot\dot{0}00\ldots qqqqq\ldots q\dot{q}_2$	See (14*h*)

7. *End Figures of P.*

Let $N=r^a+B$, where $B<r^a$.
Let P, Q be the periods of $1/N$ and $1/B$.
Then, $P\equiv Q\pmod{r^a}$, ... (15)

so that

The end group of a figures of $1/N$, $(N>r^a)$, is the *same* as the end group of a figures of $1/B=1/(N-r^a)$. ..... (15*a*)

Hence $2a$ figures of the period (P) of $1/N$ are known for all numbers $N>r^a$, viz. a ciphers at the beginning (Art. 3*a*), and the end group of a figures as above (if that group be known for the period of $1/B$); this leaves only $(\xi-2a)$ figures undetermined. Hence also the *complete period* of $1/N$ may be written down at sight for all numbers $N>r^a$ but $<r^{a+1}$, when $\xi\not>2a$, if the end group of a figures of the period of $1/B$ is known.

The Table following gives the mod 3-, or 4-figure groups of the periods of $1/N$ for all numbers (N) as follows:

4-figure endings for $r=3$; 3-figure endings for $r=5$.

This Table enables the *complete periods* of $1/N$ to be written down as follows:

In 3-scale, $N\not>122$, $\xi\not>8$; in 5-scale, $N\not>185$, $\xi\not>6$,

and part of the periods when ξ exceeds 8 or 6 respectively.

(To be continued.)

ELLIPTIC TRAMMELS AND FAGNANO POINTS.

By P. J. Harding, M.A.

1. Giulio Carlo dei Toschi di Fagnano, Count and Marquis, was an Italian mathematician of the first half of the eighteenth century. He was one of the second generation of the followers of Leibniz, a generation rather pale and unimportant compared either with the first great founders of the Differential Calculus or with the third generation that dated from about the middle of the eighteenth century, of which Lagrange was the leading genius.

Several of the men of this second generation are known mainly for some one piece of work. This is particularly the case with Fagnano, who extended the application of the Integral Calculus to the rectification of curves in a way that may be regarded as the first germ from which afterwards sprang the Theory of Elliptic Functions.

Experiencing the well known difficulty of rectifying, by elementary integration, many curves whose equations seem, at first sight, easy to deal with, he showed that on some curves two arcs could be found whose difference was rectifiable; although each separate arc presented the same difficulty as before.

Fagnano was especially successful with the Cubic Parabola, the Lemniscate and the Ellipse. He himself seems to have attached most value to his work on the Lemniscate; and at the head of his tombstone a figure of a Lemniscate was placed. But he is now better remembered in connection with the Ellipse, and certain points on this curve are associated with his name.

Fagnano Points.

2. On $AQPB$ (Fig. 1), a quadrant of a given ellipse, Fagnano found pairs of points, as P and Q, such that the difference of the arcs BP and AQ is rectifiable by ordinary integration. It was afterwards found that if perpendiculars CM and CM' are drawn from C, the centre of the ellipse, on to the tangents at P and Q, then $PM = QM'$; and that each of these is equal to the difference between the two arcs mentioned.

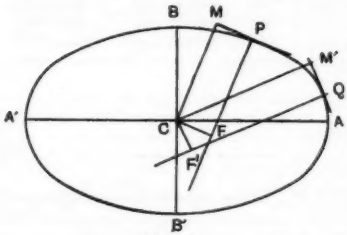


Fig. 1.

Fagnano's work, though with methods different from that of his *Produzioni Matematiche*, is considered in Williamson's *Integral Calculus* (Art. 155), and in Greenhill's *Elliptic Functions* (Art. 177). Bertrand's *Traité de Calcul Différentiel et de Calcul Intégral* is particularly clear on the subject, and a good notice of the earlier literature is to be found in Boncompagni's *Bullettino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, Vol. III.

But almost all these writers look at the question from the same point of view, and deal with lengths of arcs and with integrals that cannot be evaluated in an elementary manner; hence an impression may have arisen that there is something difficult about any work connected with such points as P and Q, and that they lie outside the range of elementary geometry. The object of this paper is to show that this is far from being the case; but that, omitting lengths of arcs and integration, very simple geometry is all that is required to obtain pairs of such points, and establish their elementary properties.

3. *Definition.* A Fagnano pair of points on an ellipse may be defined

as two points in the same quadrant for which the projections of the central radii vectores on to the tangents are equal.

Hence (Fig. 1), since $PM=QM'$, the points P and Q form a Fagnano pair.

If normals are drawn to the ellipse at P and Q, then perpendiculars, CF and CF', let fall on to them from the centre, are respectively equal to PM and QM'. Hence a Fagnano pair of points may also be defined as two points on an ellipse, the normals at which are equally distant from the centre.

This distance may be called the Fagnano distance of each point; and the semi-diameters terminating in a Fagnano pair of points may be called a Fagnano pair of semi-diameters.

It will be found that the two points of a Fagnano pair may coincide; a special point in each quadrant can thus be obtained, called in this paper the Fagnano double point and denoted by W (see Fig. 12).

N.B.—Williamson and Greenhill call this double point *the Fagnano point* of the quadrant, without giving any special name to two separate points here called a Fagnano pair.

Elliptic Trammels.

4. Elementary properties of Fagnano points are easily obtained by means of Elliptic Trammels.

In Paul Serret's *Des Méthodes en Géométrie* Trammels are used in this connection; but the following has little or nothing in common with Serret's article.

Although elliptic trammels (or elliptic compasses) are generally mentioned in books on Conic Sections, or on Practical Mechanics, their Geometry is not usually explained fully; and so, for the sake of clearness from the commencement, their use may be explained here.

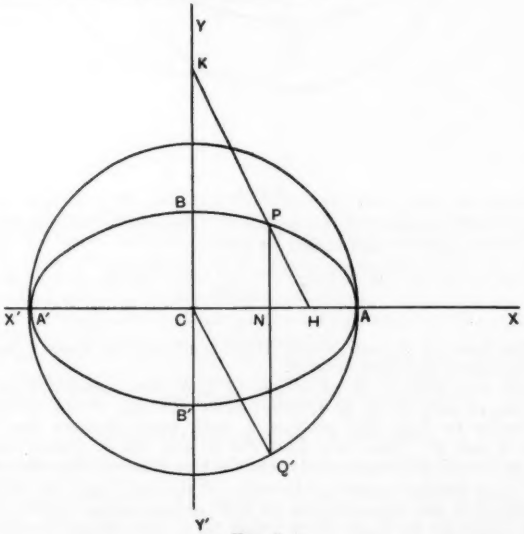


Fig. 2.

5. If H, K, P (Figs. 2 and 3) are three fixed points on a straight line or rod (fixed relatively to the rod), and if the rod slides over a fixed plane so

that the points H and K move respectively along XCX', YCY', two fixed lines in that plane perpendicular to one another, then P describes an ellipse whose axes are in the lines XX' and YY'.

We may suppose, without loss of generality, that PH is less than PK. Then there are two cases: P may be between H and K (Fig. 2) or P may be beyond H (Fig. 3). Let $PK=a$ and $PH=b$; and in each figure draw a line through C parallel to any position of HK, and another through P perpendicular to XX' and meeting in N. Let these lines meet in Q' in Fig. 2, but in Q in Fig. 3. Then $CQ'PK$ and $CQPK$ are parallelograms;

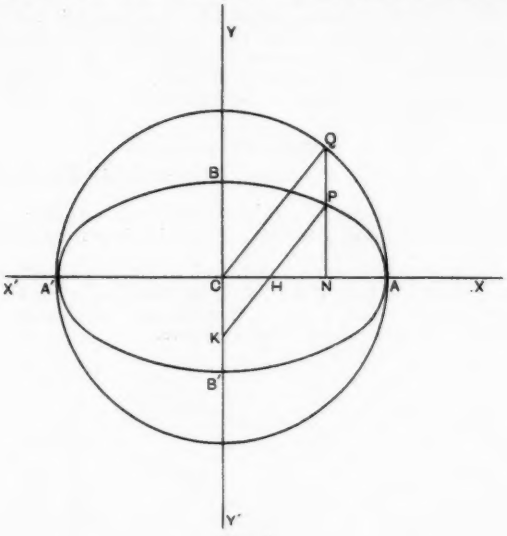


FIG. 3.

and CQ', or CQ, equals PK, *i.e.* equals a. Hence the locus of Q', or Q, is the circle with centre C and radius a; also

$$PN : NQ' \text{ (or } NQ) = PH : CQ' \text{ (or } CQ)$$
$$= PH : PK$$
$$= a : b.$$

Hence the locus of P is an ellipse whose semi-axes are a and b, and whose auxiliary circle is the locus Q or Q'.

Through any point P on an ellipse (Fig. 4) two trammel lines can be drawn, one of each kind, as follows: draw through P the ordinate PN perpendicular to XX', and produce it both ways to meet the auxiliary circle in Q and Q'. Then the lines $PH'K'$ and HPK, drawn through P parallel to CQ and CQ' respectively, are the two trammel lines through P.

6. In what follows much use is made of the fact that the inclinations (acute angles) of the trammel lines to XX', or the angles PHX' and $PH'X$, are each equal to the angle QCX; that is, to the eccentric angle of the point P.

In this paper eccentric angles will be considered simply as Euclidean acute angles; the eccentric angle of the point P will mean the acute angle

between CQ and the major axis of the ellipse. To determine completely a point on the ellipse by means of its eccentric angle, it will be necessary to state the quadrant in which the point lies, as well as the magnitude of the angle.

The eccentric angles of the extremities of conjugate diameters are in this way complementary.

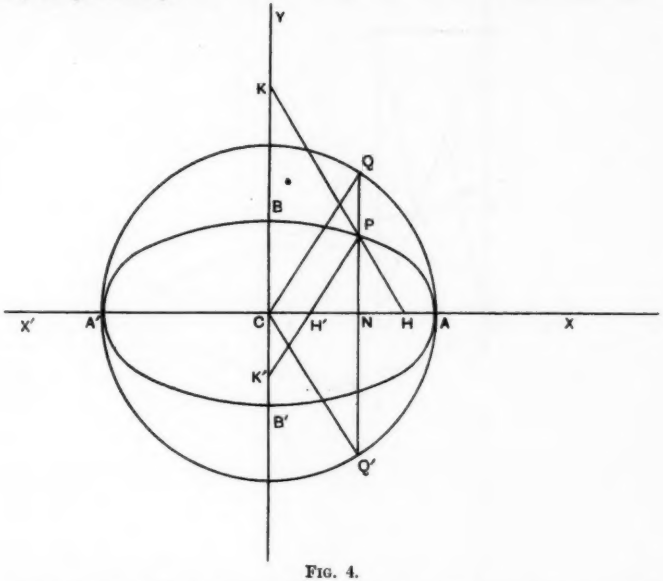


FIG. 4.

7. The lengths of the trammel lines actually used, and necessary for the description of the ellipse, are in the one case $HK = a + b$, in the other $PK' = a$. They can therefore be distinguished as the longer and the shorter trammel lines. It does not matter in most of what follows which is used; but preference will be given to the longer trammel line, which is the one that will be meant when *the trammel line* is referred to.

The point P that actually describes the ellipse, when the trammel line HK slides, will be called here "the carried point."

It is plain that if we consider symmetrical points, one in each of two adjacent quadrants of an ellipse, the longer trammel line through either will be parallel to the shorter trammel line through the other.

INSTANTANEOUS CENTRE OF ROTATION.

8. The trammel line, when moving, is in each position turning about an instantaneous centre of rotation. Consider any one position HK; then, drawing a perpendicular to XX' through H (Figs. 5 and 6) and another to YY' through K, we obtain in O, the intersection of these perpendiculars, the instantaneous centre about which the trammel line is turning.

The carried point P is, at this instant, moving in a direction perpendicular to OP. If PT is this perpendicular, then PT is the tangent, and OP is normal to the ellipse at P.

Draw CM perpendicular to PT: then PM is the Fagnano distance of P. Let CM and HK meet in Z. Then, since OP and CZ are symmetrical lines in the rectangle $CHOK$, the points P and Z are symmetrical in HK.

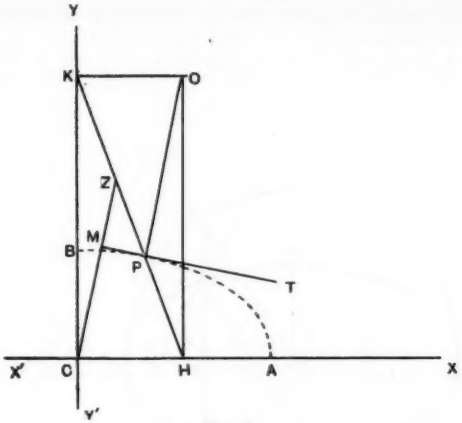


FIG. 5.

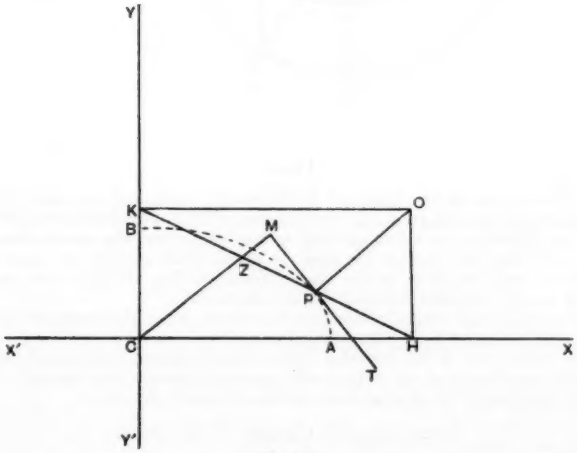


FIG. 6.

Hence $ZH=PK=a$

and $ZK=PH=b$.

In what follows considerable use will be made of the point Z, which will be referred to as the point in the trammel line symmetrical to the carried point.

Locus of Z.

9. When HK slides between XX' and YY' any point in it may be regarded as "carried point" and has an ellipse for locus.

In particular the locus of Z is an ellipse that has ZH, or a, for the length of its semi-axis in CY, and ZK, or b, for the length of its semi-axis in CX.

Let the two ellipses, that are the loci of P and Z (Fig. 7), be distinguished

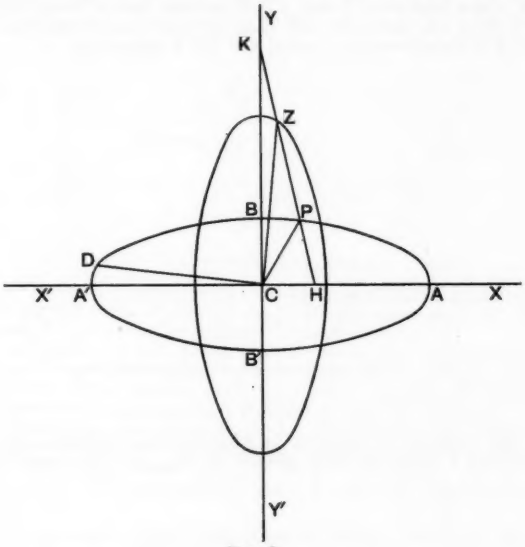


Fig. 7.

as the P-ellipse and the Z-ellipse. They are identically equal; and, by turning the P-ellipse about C through a right angle, it will be brought into complete coincidence with the Z-ellipse.

Before this rotation takes place, consider any one position of HK with P and Z marked on it; and draw CD a semi-diameter of the P-ellipse perpendicular to CZ.

Then, by a rotation of the P-ellipse that brings it into coincidence with the Z-ellipse, CD can be made to coincide with CZ.

Hence CD equals CZ.

But CZ is perpendicular to the tangent to the P-ellipse, at P. Therefore CD is parallel to that tangent; or CP and CD are conjugate semi-diameters of the P-ellipse.

Now, omitting the Z-ellipse from further consideration, we see that, as HK slides between XX' and YY' carrying P and Z with it, CZ is always equal to the semi-diameter conjugate to CP in the elliptic locus of P.

Changes in the Fagnano Distance.

10. The changes in the Fagnano distance of any point on an ellipse, as it moves over the quadrant BA, can easily be seen, without any further reference to the centre of instantaneous rotation, by watching the motion

of the trammel line HK, on which are marked the carried point P and its symmetrical Z.

In whatever position HK may be (see Figs. 8, 9 and 10), join CZ, producing it if necessary, and draw PM perpendicular to it.

Then PM, the Fagnano distance of P, is in general one side of the right-angled triangle PMZ, and is less than PZ, *i.e.* less than $a-b$.

Let HK start from coincidence with CY, H being at C. Then PM is zero.

As HK slides between CX and CY, H moving from C along CX, and K towards C along CY, the angle CZH is at first acute and increasing (Fig. 8). The point M is then between C and Z, and PM is increasing.

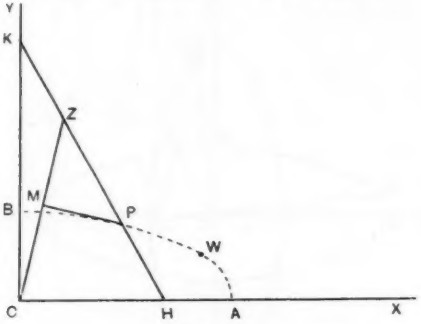


FIG. 8.

When HK reaches the position in which the angle CZH is right, the point P reaches a special position marked W (Fig. 9). The line HK is then

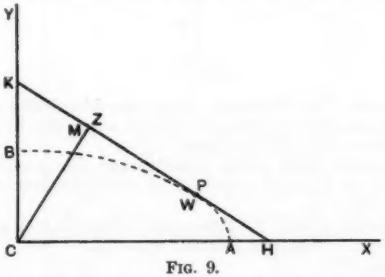


FIG. 9.

a tangent to the ellipse at W, for M coincides with Z: PM is a maximum and equal to $a-b$.

As the movement of HK continues, the angle CZH becomes obtuse (Fig. 10). The point M is on CZ produced: PM is again less than $a-b$, but is now decreasing.

Finally, when HK is coincident with CX, K being at C, PM returns to its initial value zero.

Thus, as P moves over the arc BW, its Fagnano distance increases from zero to $a-b$: as it continues its movement over the arc WA, its Fagnano distance decreases from $a-b$ to zero. Therefore, any point P_1 in BW must have corresponding to it a point P_2 in WA, so that the Fagnano distances

of P_1 and P_2 are equal; that is, so that P_1 and P_2 form a Fagnano pair (Fig. 12).

When P_1 is at B, P_2 is at A. Remaining a Fagnano pair, P_1 and P_2 move over the arc BA each towards W, and finally coincide in W, the Fagnano double point.

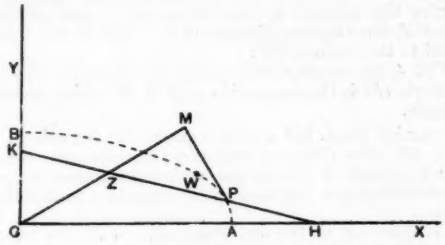


FIG. 10.

TRAMMEL TRIANGLES.

11. Any position, between given rectangular axes, of the trammel line HK, with the carried point P marked on it, determines a point on the ellipse whose axes of figure coincide with the rectangular axes.

Now the position of HK between the axes is completely determined by the intercepts CH and CK; that is, by the triangle CHK. Hence this triangle, in itself, determines a definite point in an indicated quadrant of the ellipse, and will be called in this paper *the trammel triangle* of that point.

N.B.—It is to be understood that when the trammel line HK is referred to, the carried point P is always supposed to be marked on it; and this whether HK is considered alone, or as the hypotenuse of the trammel triangle CHK. The symmetrical point Z may also be supposed always marked on HK.

12. The rectangular axes being given in position and fixed, the trammel triangle of any point may be supposed moved to, or redrawn in, any other part of the plane. It will remain a representative of the point. For to obtain this point from the new position, the triangle has only to be moved back and fitted properly between the axes,—it may have to be turned over for this,—and the carried point will coincide with the actual point on the ellipse represented by the triangle.

In figures representing a trammel triangle away from the axes (see Fig. 11) points will be represented by small letters corresponding to capital letters used when the triangle is in its original position.

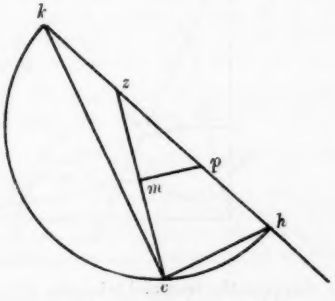


FIG. 11.

Thus, draw in any convenient position a line hk whose length $=a+b$ (Fig. 11). Mark the point p on it so that $pk=a$ and $ph=b$. Describe a semicircle on hk, and take any point c on its circumference. Then the triangle chk, with p as carried point, is the trammel triangle, and representative, of some point P on the ellipse whose semi-axes are equal to

a and b; and P will be the position of p when the triangle is fitted back, or redrawn, in the angle XCY, so that ch and ck lie along CX and CY respectively.

13. If z is the point symmetrical to p in hk, and pm perpendicular to cz, and if these lines are carried back with the triangle chk, the new position of pm will give the tangent to the ellipse at P, and in length pm will coincide with PM, the Fagnano distance of P. Also cz will coincide with CZ and be parallel to the normal at P.

Since, in Fig. 4, the angle CHK equals the eccentric angle of P; so, in Fig. 11, the angle chk is the eccentric angle of the point represented by the trammel triangle.

14. If hk remains fixed, but c moves round the semicircle, the different positions of c will give different trammel triangles each representing one point on the quadrant of the ellipse under consideration; or, there is a one-to-one correspondence between points on the semicircle and points on the quadrant.

Of course, it does not matter on which side of hk the trammel triangle is drawn; provided it is understood that on fitting back chk between CX and CY, ch is made to lie along CX and ck along CY. Therefore, if a complete circle is described on hk as diameter, there are on it two possible positions for c, such that chk will be the trammel triangle representing a definite point P on the ellipse:—two positions that are the two ends of a chord of the circle perpendicular to hk.

Trammel Quadrilaterals.

15. Let P_1 and P_2 be two points on an ellipse, H_1K_1 and H_2K_2 the trammel lines passing through them (Fig. 12), and Z_1, Z_2 the points on those lines symmetrical to P_1 and P_2.

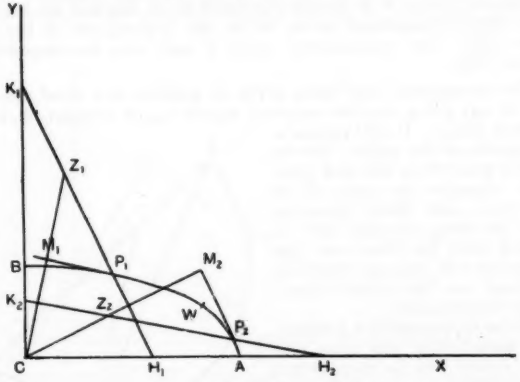


Fig. 12.

Suppose the trammel triangles CH_1K_1 and CH_2K_2 to be moved to another part of the plane and placed down again so that their hypotenuses coincide.

Let h be the new position of both H_1 and H_2 (Fig. 13), k the new position of K_1 and K_2; then P_1 and P_2 will coincide at a point in hk, say p; and Z_1, Z_2 will coincide in a point z.

For clearness of figure, and, it will afterwards be seen, for simplicity of theorem, let the triangles be placed on opposite sides of hk; so that c_1hk is

the new position of CH_1K_1, and c_2hk of CH_2K_2. There will thus be formed a quadrilateral c_1hc_2k, which, by means of its two trammel triangles, will be a representative of the two points P_1 and P_2; and which, if P_1 and P_2 were not previously known, would enable us to mark their positions relatively to the axes.

This quadrilateral will be called the *Trammel Quadrilateral* of the two points.

In what follows, the points represented by a trammel quadrilateral will generally be taken both in the first quadrant; but this is not necessary. To determine the two points in any case, besides being given their trammel quadrilateral, the quadrant in which each lies must be indicated.

The line hk, with the points p and z on it, must be regarded as belonging to each of the two trammel triangles that compose the quadrilateral. We may call hk the principal diagonal of the quadrilateral; and the points p, z may still be called the carried point and its symmetrical.

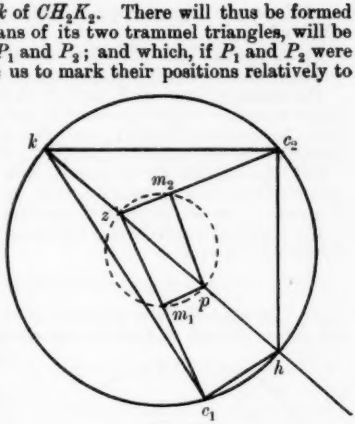


FIG. 13.

16. Since in Fig. 12 the lines P_1M_1 and P_2M_2, drawn perpendicular to CZ_1 and CZ_2, are the Fagnano distances of P_1 and P_2, and in direction give the tangents to the ellipse at those points; so, in the trammel quadrilateral, if c_1z and c_2z are joined and pm_1, pm_2 drawn perpendicular to them, we see that pm_1, pm_2 are the Fagnano distances of P_1 and P_2; the angles zpm_1 and zpm_2 are equal to the angles between trammel line and tangent at those points; and the distances of the centre of the ellipse from the tangents at P_1 and P_2 are equal to c_1m_1 and c_2m_2.

The angles c_1hk and c_2hk, being the new positions of CH_1K_1 and CH_2K_2, are equal to the eccentric angles of P_1 and P_2.

17. The lines pm_1 and pm_2 are, of course, chords of the circle described on pz as diameter.

If W is the Fagnano double point, and if P_1 and P_2 both lie in BW, then M_1 is between C and Z_1, and M_2 between C and Z_2. Hence also m_1 and m_2 have corresponding positions in c_1z and c_2z; and therefore pm_1, pm_2 will be on opposite sides of pz.

Similarly, if P_1 and P_2 both lie in WB, the points M_1 and M_2 will lie in the productions of CZ_1 and CZ_2, and again pm_1 and pm_2 will lie on opposite sides of pz.

But if P_1 and P_2 are one in BW and the other in WA, it will be seen that pm_1 and pm_2 are both on the same side of pz.

It is obvious that Figs. 12 and 13 are not constructed to correspond exactly; but that in Fig. 13 we have the trammel quadrilateral of two points both in BW; additional figures can easily be made if required.

18. The trammel quadrilateral representing any two points on an ellipse is obviously inscribable in a circle.

Hence to construct a trammel quadrilateral for any pair of points: first draw in any convenient position hk equal to the sum of the semi-axes of the ellipse, and mark p and z as before. Describe a circle on hk as diameter, and take two points c_1 and c_2 on its circumference one on each side of hk. Then c_1hc_2k is a trammel quadrilateral representing two points on the ellipse; and by varying c_1 and c_2, or by taking them to satisfy certain other

conditions, the trammel quadrilateral representing any two points can be obtained.

19. A special pair of points on the ellipse will be represented by a trammel quadrilateral of a special nature.

For example, suppose P_1 and P_2 to be extremities of conjugate diameters. Their excentric angles (measured each as an acute angle, see Art. 6) will then be complementary; and therefore in the corresponding trammel quadrilateral the angle c_1hc_2 will be a right angle.

But the angles at c_1 and c_2 are right angles. Hence the trammel quadrilateral is a rectangle; that is, it is a quadrilateral whose second diagonal c_1c_2 passes through the middle point of the principal diagonal hk.

The trammel quadrilateral representing the extremities of equi-conjugate diameters is a square.

Fagnano Trammel Quadrilateral.

20. The characteristic property of the trammel quadrilateral c_1hc_2k, when the points P_1 and P_2 that it represents form a Fagnano pair, may now be easily seen.

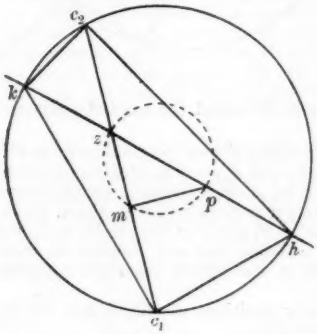


Fig. 14.

For, in this case, pm_1 and pm_2 are two equal chords of the circle on pz; and they lie on the same side of pz, since P_1 is in BW, but P_2 in WA. Hence they coincide entirely. See Fig. 14, where the common position of m_1 and m_2 is denoted by m.

Therefore, when c_1hc_2k represents a Fagnano pair of points, the lines c_1z and c_2z are perpendicular to the same line pm; or, in this case, c_1zc_2 is a straight line.

The result just reached is the central theorem of this paper, and may be enunciated generally as follows:—When a trammel quadrilateral represents a Fagnano pair of points, its second diagonal cuts its principal diagonal in the point symmetrical to the carried point.

Such a trammel quadrilateral will be called a *Fagnano trammel quadrilateral*; or, shortly, a *Fagnano quadrilateral*.

21. When a Fagnano quadrilateral c_1hc_2k (Fig. 14) is separated into its two trammel triangles, and these are fitted back, or redrawn, between CX and CY, the two new positions of p are the Fagnano pair of points P_1 and P_2 (Fig. 12) represented by the quadrilateral.

If the triangle c_1hk carries back with it c_1z and pm, and the triangle c_2hk carries with it c_2z and pm, the two new positions of pm will give in direction the tangents to the ellipse at P_1 and P_2, and in length the equal Fagnano distances P_1M_1 and P_2M_2.

Clearly also $c_1m = CM_1$ and $c_2m = CM_2$.

If we suppose added to Fig. 12 the semi-diameters CD_1 and CD_2 conjugate to CP_1 and CP_2, as in Fig. 15, we shall have (Art. 9)

$$c_1z = CZ_1 = CD_1$$

and

$$c_2z = CZ_2 = CD_2.$$

The angle mpz in Fig. 14 is equal to the angle between the tangent and trammel line at each of the points P_1 and P_2. Also the angle mzp equals the angle between the normal and trammel line at each of these points.

(To be continued.)

THE TEACHING OF LIMITS AND CONVERGENCE TO SCHOLARSHIP CANDIDATES.

I.

1. The present paper is an attempt to sketch a mode of procedure for teaching Limits and Convergence to beginners. It is felt by many teachers that at present

(1) the subject is introduced too academically ;

(2) that its Arithmetical foundation is not brought into sufficient prominence ;

(3) that therefore pupils leave school with only a very hazy notion of what it is all about ;

(4) that the investigation of the Convergence of such things as $\Sigma \frac{n^n x^n}{\lfloor n}$

filling up page after page of our Algebras is rather profitless and barren of ideas.

It is suggested that the subject should be introduced to the beginner in pretty much the same way as a teacher of Natural History introduces Botany. He makes his pupils examine individual plants, and points out their most obvious properties without attempting at first any systematic classification or process of reasoning. In this way the pupil will gain in the case of Convergence, as of Botany, some practical first-hand acquaintance with the phenomena to be looked for, and the use of the subject as a practical instrument. Then, and not till then, he will be in a position to appreciate the principles underlying the rigorous academic proofs.

2. One might introduce the subject by pointing out that in every occupation of life, men are continually engaged in arranging things in sequences obeying some fixed law. Thus in a class the master arranges his pupils in order according to their intellectual power in some given subject, *e.g.* Classics. If they also be arranged according to their mathematical abilities a different order or sequence will be obtained. The things to be arranged are the same in both cases, but the law defining the order or sequence is not.

Consider the sequence of natural numbers

$$1,\ 2,\ 3,\ 4,\ 5,\ 6, \ldots$$

as far as we care to go.

There is one fundamental difference between the sequence involving the natural numbers and the former sequences involving the pupils. In the first case, the number of pupils at our disposal is limited ; in the second case there is no limit to the number of integers at our disposal.

We can construct sequences according to very many laws, and we shall use the natural numbers to label for purposes of identification the successive members of the sequence.

Thus, suppose the sequence is that of the Prime Numbers, we obtain the following chart which may be continued as far as we please :

Prime Numbers									
Label,	-	1,	2,	3,	4,	5,	6,	7,	8 ...
Sequence,		1,	2,	3,	5,	7,	11,	13,	17 ...

The Arithmetical Progression beginning with 3 and having 5 as the common difference may be tabulated thus :

A.P. beginning with 3 and having 5 as common difference.

Label,	-	1,	2,	3,	4,	5,	6,	7,	8,	9,	10, ...
Sequence,		3,	8,	13,	18,	23,	28,	33,	38,	43,	48, ...

3. We shall next consider some practical uses to which a sequence can be put.

To approximate in decimal form to the positive square root of 2.

Since $\sqrt{2}\times\sqrt{2}=2$, $1\times1=1$, and $2\times2=4$,

we conclude that $\sqrt{2}$ lies between 1 and 2. This gives the first step in the sequence.

Again by trial $1{\cdot}4\times1{\cdot}4=1{\cdot}96$ and $1{\cdot}5\times1{\cdot}5=2{\cdot}25$.

Hence we conclude that $\sqrt{2}$ lies between 1·4 and 1·5. This gives the second step in our sequence.

Further, by trial of 1·40, 1·41, 1·42, 1·43, ... 1·49,

we obtain $1{\cdot}41\times1{\cdot}41=1{\cdot}9881$, $1{\cdot}42\times1{\cdot}42=2{\cdot}0164$,

from which we conclude that $\sqrt{2}$ lies between 1·41 and 1·42.

Thus by a perfectly definite law, whereby each member of the sequence can be found from those that precede it, we can proceed by the method of sequences to find a succession of pairs of decimals, ever getting closer and closer, between which $\sqrt{2}$ lies. We can exhibit these sequences on a chart thus:

Sequences approximating to $\sqrt{2}$.

Label, - - - - - - -	1,	2,	3,	4	... etc.
Lower Sequence, - - - - -	1,	1·4,	1·41,	1·414	... etc.
Upper Sequence, - - - - -	2,	1·5,	1·42,	1·415	... etc.
Squares of Members of Lower Sequence,	1,	1·96,	1·9881,	1·999396	... etc.
Squares of Members of Upper Sequence,	4,	2·25,	2·0164,	2·002225	... etc.

4. Many further points are to be noted in the above calculation. Suppose, for example, we wished to carry on the process until the members of either the upper or the lower sequence differed from $\sqrt{2}$ by less than some given quantity, say ·001.

If the members of the lower sequence be denoted by $l_1, l_2, l_3 \ldots$,
and those of the upper sequence be denoted by $u_1, u_2, u_3 \ldots$,
we wish to find the indicator number n such that

$$\sqrt{2}-l_n \leqq \cdot001$$

and $$u_n-\sqrt{2} \leqq \cdot001.$$

Now since $\sqrt{2}$ always lies between corresponding members of the upper and lower sequences, it will be sufficient for our purpose to go on with the process until the gap between corresponding members of the upper and lower sequences be $< \cdot001$. Then $\sqrt{2}$ would differ from either by some quantity $< \cdot001$.

Corollary. If l_n differ from $\sqrt{2}$ by some quantity $< \cdot001$, it is evident that l_{n+1}, l_{n+2}, etc., all differ from $\sqrt{2}$ by some quantity $< \cdot001$.

For l_{n+1} is greater than l_n since a decimal place has been added on to l_n to obtain l_{n+1}, and furthermore all the members of the lower sequence are each $< \sqrt{2}$ by the nature of the case.

A similar result holds with regard to u_n, u_{n+1}, u_{n+2}, etc.

The tabulation of the gaps u_1-l_1, u_2-l_2, etc., is of interest.

Label, - -	1,	2,	3,	4,	etc.
Gap Sequence,	1,	·1,	·01,	·001,	etc.

Also, if we remember that 2 lies between u_n^2 and l_n^2 the gap sequence

$$u_1^2-l_1^2,\ u_2^2-l_2^2,\ u_3^2-l_3^2 \ldots$$

is instructive in shewing the rapidity of the approximation.

Label, - - -	1,	2,	3,	4,	etc.
Gap Sequence $u_n^2 - l_n^2$,	3,	·29,	·0283,	·002829,	etc.

5. Exactly the same process can be used to find by a sequence approximate values for the cube root or the fifth root, etc., of any number.

6. *Sequence for an equation.*

Consider the equation $x^2 - 7x - 4 = 0$.

We have to find values for x such that y defined by

$$y = x^2 - 7x - 4$$

shall vanish.

Graphs shew that one root of the above equation is positive and the other negative. Trial shews that $x=7$ renders y negative and $x=8$ renders y positive. Our object is to find a value for x that shall make y vanish. We therefore construct by trial a pair of sequences, the corresponding members of which are consecutive decimals, and such that the members of the lower sequence render y negative when substituted for x, while the members of the upper sequence render y positive.

Label, - -	1,	2,	3,	4,	etc.
Values of x making y $-^{ve}$, L	7,	7·5,	7·53,	7·531,	etc.
Values of x making y $+^{ve}$, U	8,	7·6,	7·54,	7·532,	etc.
Result of L-substitution,	−4,	−·25,	−·0091,	−·001039,	etc.
Result of U-substitution,	4,	·56,	·0716,	·007024,	etc.

7. *Divergent Sequences.*

The subject of divergent sequences is not very difficult for the beginner to grasp, and can be introduced by the following illustration, which (unconsciously or not) we frequently apply to the affairs of everyday life.

Suppose that the inhabitants of some town wished to erect a statue to Sir Walter Scott at a cost of £2000. They were, however, unable to collect more than £1500, and they decided to let it lie in the bank at Compound Interest till it amounted to the sum required for the erection of the statue.

This leads at once to a sequence exhibiting the chief fundamental properties of divergent sequences.

Suppose the rate of Interest to be 5 % per annum. We have, on calculation, the following chart:

Year, - - -	1,	2,	3,	4,	
Sum deposited at beginning of each year,	1500,	1575,	1653·75,	1736·4375,	
Year, - - -	5,		6,	7,	etc.
Sum deposited at beginning of each year,	1823·259375,		1914·42234375,	2010·1434609375,	etc.

Thus we see that there will be sufficient money to erect the statue after six years.

The following points should be noted about the above process:

I. A definite law is given whereby the sequence is defined.

II. We may see *intuitively* that we can proceed until on and after a certain year the sum deposited exceeds any sum that is named, *e.g.* £3000, £4000 or £100,000. This of course requires proof.

8. In many cases the law defining the sequence is given in a more academic form.

Suppose that any three consecutive members of a sequence are connected by the following law :

$$p_n = 4p_{n-1} + 5p_{n-2}.$$

Then it is plain that if we be given the first two members of the sequence, all subsequent members can be computed successively by the above law of formation. Thus suppose that the first two members are 2 and 3. We then have the following chart :

Chart of sequence defined by $p_n = 4p_{n-1} + 5p_{n-2}$ where $p_1 = 2$ and $p_2 = 3$.

Label,	1,	2,	3,	4,	5,	etc.
Sequence,	2,	3,	22,	103,	522,	etc.

Here again it can be *intuitively* perceived that (1) there is a definite law defining the sequence, and (2) that we can find a definite member of the sequence, such that it and all subsequent members are greater than any definite number we choose to name, *e.g.* 300, 500 or 900,000.

9. Hitherto, we have devoted our attention to numerical calculation and to the indication and *intuitive* perception of such phenomena as presented themselves. This, however, is not sufficient for rigorous reasoning, and one or two proofs of a more general nature might here be given.

Example 1. *To prove that it is possible to find a member of the sequence*

$$12^1,\ 12^2,\ 12^3, \ldots 12^n, \ldots$$

such that it and all subsequent members are greater than any definite number we choose to name.

For example, let the given number be 958463.

Then $958463 < 10^6$ and $12^6 > 10^6$;

$\therefore$ the sixth member and all subsequent members of the above sequence are > 958463.

Otherwise. $12^n = (1+11)^n$, (n being supposed a $+^{ve}$ integer)

$> 1 + 11n$, (by the Binomial Theorem) ;

$\therefore\ 12^n > 958463$,

if $1 + 11n > 958463$,

i.e. if $11n > 958462$,

i.e. if $n > 87132\frac{10}{11}$.

Hence the 87133rd member of the above sequence and subsequent members are > 958463.

N.B.—The first solution plainly gives the more convenient answer.

Example 2. *To shew that, given a sufficient time, any sum of money collected* (e.g. £1500) *and invested at Compound Interest* (e.g. *at* 5 % *per annum*) *is sufficient to erect a statue of any cost* (e.g. £3000).

The sum deposited amounts after n years to $1500 \times 1{\cdot}05^n$ which is > 3000 if $1{\cdot}05^n > 2$.

i.e. if $(1 + {\cdot}05)^n > 2$,

i.e. if $1 + {\cdot}05n > 2$; $\because\ (1 + {\cdot}05)^n > 1 + {\cdot}05n$, (Binomial Theorem),

i.e. if $n > 20$.

N.B.—It is not stipulated that the above methods give the *earliest* member of the sequence that exceeds the given quantity, but only that they give *a* member greater than the given quantity.

Example 3. This example is of a more academic type.

To find a member of the sequence

$$\frac{1}{1},\ \frac{1}{1}+\frac{1}{2},\ \frac{1}{1}+\frac{1}{2}+\frac{1}{3},\cdots\ \frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{n},\cdots$$

such that it and all subsequent members are greater than any assigned number (e.g. 1000).

Any positive integer lies between two consecutive powers of 2, *e.g.* 41 lies between 2^5 and 2^6. Let n lie between 2^κ and $2^{\kappa+1}$.

Then $$\sum_1^n \frac{1}{m} > 1000 \quad \text{if} \quad \sum_1^{2^\kappa} \frac{1}{m} > 1000.$$

Now

$$1 = 1$$

$$\tfrac{1}{2} = \tfrac{1}{2}$$

$$\tfrac{1}{3}+\tfrac{1}{4} > \tfrac{1}{4}+\tfrac{1}{4}, \qquad \textit{i.e.}\ \tfrac{1}{2},$$

$$\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8} > \tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}, \qquad \textit{i.e.}\ \tfrac{1}{2},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{1}{2^{\kappa-1}+1}+\frac{1}{2^{\kappa-1}+2}+\ldots\frac{1}{2^\kappa} > \frac{1}{2^\kappa}+\frac{1}{2^\kappa} \text{ to } 2^{\kappa-1} \text{ terms}, \quad \textit{i.e.}\ \tfrac{1}{2};$$

$$\therefore \sum_1^{2^\kappa} \frac{1}{m} > 1+\frac{\kappa}{2}.$$

Hence $$+\sum_1^{2^\kappa} \frac{1}{m} > 1000 \quad \text{if} \quad 1+\frac{\kappa}{2} > 1000, \quad \textit{i.e.}\ \text{if} \quad \kappa > 1998,$$

whence the result is evident.

10. At first sight the following might appear a feasible method of taxing a community so as to obtain a sufficient sum of money for the erection of a Town Hall.

Suppose that £3000 *is wanted and that the inhabitants bind themselves to supply* £1000 *the first year and to halve their donations every successive year until the required sum of* £3000 *be reached.*

The donations and the sums in hand can be exhibited as sequences in the following chart:

Year, - - -	1,	2,	3,	4,	5,	6,	etc.
Donation, - -	1000,	500,	250,	125,	62·5,	31·25,	etc.
Total sum collected,	1000,	1500,	1750,	1875,	1937·5,	1968·75,	etc.

The chart might lead us to expect that we should before very long reach the required sum of £3000, and we thus learn that the *intuitive* method of viewing the charts of sequences, though frequently guiding us, may often lead us astray unless we are careful to verify by exact calculation. As a matter of fact we can never, by the above process of taxation, reach £3000.

For the sequence to be considered is as follows:

$$1000,\quad 1000\left(1+\frac{1}{2}\right),\quad 1000\left(1+\frac{1}{2}+\frac{1}{2^2}\right),\quad \ldots,\quad 1000\left(1+\frac{1}{2}+\ldots\frac{1}{2^n}\right),\quad \ldots,$$

i.e. by the ordinary method of calculating a Geometrical Progression,

$$1000,\quad 1000\frac{1-\frac{1}{2^2}}{1-\frac{1}{2}},\quad 1000\frac{1-\frac{1}{2^3}}{1-\frac{1}{2}},\quad \ldots,\quad 1000\frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}},\quad \ldots,$$

i.e.

$$2000\left(1-\frac{1}{2}\right),\quad 2000\left(1-\frac{1}{2^2}\right),\quad 2000\left(1-\frac{1}{2^3}\right),\quad \ldots,\quad 2000\left(1-\frac{1}{2^{n+1}}\right),\quad \ldots,$$

whence it is plain that all the members of the above sequence are less than 2000, however far we like to go. It is therefore impossible by this method of procedure for the inhabitants of that community ever to reach £3000.

I hope to deal with the Teaching of Convergent Sequences in a subsequent paper, and meanwhile beg to express my indebtedness to Dr. Charles M'Leod, Aberdeen Grammar School, for criticising the present article while in manuscript form.

W. P. MILNE.

ANALYSIS AND PROJECTIVE GEOMETRY.

FEW branches of elementary mathematics have escaped reform in recent years; algebra, trigonometry, elementary geometry, statics, and the calculus have all been transformed to suit the practical needs of the time. But amid all this change, there have been few attempts to alter the methods employed in introducing students to the geometry of the conic. Dr. Filon, in his most interesting treatise on Projective Geometry published in 1909, points out that students learn the same facts about the conic practically three times over, (1) analytically, (2) through a course of what is called "geometrical conics" based on the focus-directrix definition, and (3) projectively; and his book indicates a method of co-ordinating and uniting the last two systems of approach. Although it is undoubtedly the general custom in this country to start in the first place from the focus definition, it is worth while considering afresh whether this plan is really the most advantageous.

It is interesting to note that historically this was not the starting point of the early geometers. It is true that the focus-directrix property was known to Pappus, but so far from being regarded as fundamental, it was actually lost sight of altogether, until attention was called to it by Newton. The geometrical investigations of Apollonius were based on the conception of the conic as the plane section of a cone, the advantages of which plan may be enumerated roughly as follows:

In the first place, by the aid of a model it is easy to show the shapes of the different species of conics that arise, the nature of the transition from one to the other, and their relation to and dependence upon the circle. And this emphasises the sense of continuity which should link up the student's knowledge of circle-properties to the corresponding generalised conic-properties, and enables him to extend to the conic with valuable rapidity a whole class of properties of the circle, with which he is already familiar. To take a simple example, the existence of a diameter of a conic seems a mere accident when evolved from the focus definition, but under a projective treatment such an idea, so far from appearing a happy chance, is seen to be part of the logical harmony of the subject—and this with still greater clearness, if by the aid of analysis the methods of orthogonal projection are introduced at the start.

And then, again, a factor of considerable importance is the novelty of idea with which the student is faced. The ordinary course of geometrical conics provides excellent practice in rider work, and may well supply a systematic acquaintance with the metrical theorems, but the methods employed are those to which he is already accustomed, and few, if any, of the ideas it yields are of a new character. Whereas the projective treatment opens out to view a region of geometrical thought unlike anything that has been met with in the past, and so not only arouses interest, but also may be justly regarded as of real educational value. The conception of imaginary elements and the proper appreciation of the meaning of ideal points or points at infinity and the

applications of general projection are stimulating and fruitful; and, at the same time, the student's vanity is flattered by the gift of a weapon which enables him to discover and invent all kinds of complicated properties for himself.

On all these grounds it may be urged that projective methods should be introduced at as early a stage as is practicable. But, of course, it may be said that the difficulties and novelties of the subject require a definite maturity of mind which is only attained by students towards the end of their school course. The object of this paper is to argue that a judicious intermixture of geometrical and analytical methods will do much to smooth the path and enable the subject to be started as soon as a simple acquaintance with the analytical geometry of the circle and its geometrical harmonic properties has been attained, and before the formal course of geometrical conics is taken. It is therefore desirable to summarise the reasons why it is advantageous to employ analysis from the start.

By building upon an analytical foundation, the introduction of imaginary and ideal elements follows with natural ease from the elementary principles of algebra. The idea, for example, that any line meets any circle at two points, real or imaginary, conveys no meaning to the student, unless he has in his mind an analytical definition of what constitutes a circle; but, as soon as he looks at it through analytical glasses, it presents no difficulties; and the first step has been taken towards an understanding of the true bearing of the Principle of Continuity. For the average student, this Principle must be merely an interpretation of an analytical phenomenon, and such applications of it, as are made, are most easily grasped if he keeps the analytical aspect always in front of him. Otherwise there is a very real danger that it never becomes more than a vague and undefined privilege, which appears to be claimed for the purpose of abbreviating proofs of certain general theorems.

The desirability of expounding the principles of general projection, for which imaginary elements are required, has been already noticed. No one can help being interested by the peculiar properties of the circular points and isotropic lines. There is so much that at first sight seems anomalous, the absence of any sense of direction and curious metrical effects all combine to cause both amusement and curiosity. And analytical methods are peculiarly suited for the investigation of these phenomena.

But another consideration of greater importance is the fact that if analysis is discarded, it is necessary to postpone the treatment of imaginary projection until after a study of involution. Prof. Lodge, at the general meeting of the Mathematical Association in 1909, pointed out the advisability of introducing a simple course on involution at an early stage. Whether or not this is adopted, it is at least probable that analytical methods would still further elucidate the difficulties of the subject. But in any case the *definition* of a conjugate point-pair as the double points of an overlapping involution is likely to give an air of artificiality to the whole theory, and to convey, at any rate to the average student, no meaning whatever. And this all combines to convince him that the subject is little more than a complicated conjuring-trick, which he will never be able to perform himself. But a still further disadvantage lies in the fact that the theory of foci must also be postponed till after a course of involution, or else must be developed (projectively) in a one-sided manner, which is out of keeping with the other broad conceptions of the subject. The *definition* of the foci as the meets of isotropic tangents is so fruitful a view-point, and may be made the base

of so symmetrical a development, that it is most desirable that it should be introduced at the earliest opportunity.

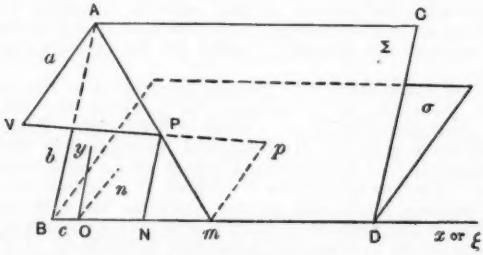
The purport of this argument will be rendered clearer by the addition of a few details as to an analytical treatment of the kind mentioned above, which has been in use for a few years.

To start with, it is necessary to provide definitions which cover the case of imaginary elements, such as the point, line, angle and cross-ratio: and for this object, it is well to take the analytical results, *proved* for *real* elements, and embody them in a definition for imaginary elements, *e.g.* the cross-ratio of the lines $y=m_1x$, $y=m_2x$, $y=m_3x$, $y=m_4x$, some of which are imaginary, would be *defined* as $\frac{(m_1-m_2)(m_3-m_4)}{(m_1-m_4)(m_3-m_2)}$, etc. It is then easy to prove a number of simple but important geometrical properties, as for example Laguerre's theorem that if α is the angle between the lines $y=mx$, $y=m'x$, then the cross-ratio of the four lines $y=mx$, $y=ix$, $y=m'x$, $y=-ix$ is $\exp(2\alpha i)$, and deduce several important results, such as the fact that any pair of perpendicular lines are harmonically conjugate to the isotropic lines.

The next step is to introduce the conception of ideal points. If this is done by identifying every system of parallel lines with an ideal point and the aggregate of such points with the line at infinity, an analytical representation may then be effected as follows:

Take $\frac{x}{z}$ and $\frac{y}{z}$ as the Cartesian coordinates of a point, where x, y, z are connected by the invariable relation $x+y+z=1$; by considering the equations of two or more parallel lines, it then appears that $z=0$ is to be regarded as representing the line at infinity and it follows immediately that any circle $x^2+y^2+2gxz+2fyz+cz^2=0$ meets this line at two points, ω, ω' say, which are the same for all circles and that any curve of the second degree through ω, ω' must be a circle. It is scarcely necessary to add that a considerable amount of general discussion is needed before the class understands the real significance of these points and their hypothetical nature.

To open the way to the conic, it is next necessary to obtain a relation between the coordinates of a point and those of its conical projection.



Let P be any point in the plane Σ and let p be its projection on the plane σ with respect to the vertex of projection V; BD is the line of intersection of Σ and σ; a plane through V, parallel to σ, meets Σ in the line AC; a plane through V perpendicular to BD meets AC, BD in A, B; AP meets BD in m and the perpendicular from P to BD meets it in N. Any point O on BD is taken as origin, and OD is the x-axis for Σ and the ξ-axis for σ, and the coordinates of P and p referred to these systems of axes are respectively x, y and ξ, η. The position of V is determined by the following constants: $VA=a$, $AB=b$, $BO=c$. By using the similar triangles

VAP, pmP and the fact that PN, AB are parallel, it is easy to show that

$$x=\frac{a\xi-c\eta}{\eta+a}, \quad y=\frac{b\eta}{\eta+a}.$$

It is of course obvious that this relation provides short proofs of a number of projection-theorems and may be used to establish rigorously the validity of imaginary projection. It is easy to find for instance a simple transformation of this kind which turns the general equation of the second degree into that of a circle, or one which will project any two given points into ω, ω'; for example, the relation $\xi=\frac{gx}{y-g}$, $\eta=\frac{ify}{y-g}$ projects the points (f, g), $(-f, g)$ into ω, ω'.

In the final stage, after defining the conic as the projection of a circle and giving the usual projective definitions of the centre, asymptotes, parabola, etc., it may then be shown that every conic is represented by an equation of the second degree and that every conic through ω, ω' is a circle and that $C\omega$, $C\omega'$ are the asymptotes of any circle, centre C. The foci are then defined as the meets of the tangents from ω, ω' to the conic and their properties and also those of confocal conics follow easily from the harmonic properties of the circumscribing quadrilateral.

Although no doubt a considerable amount of rider-work will be essential, this course now leaves the student in the position of being able to use to the fullest degree every form and variation of general projection. And by its aid, many of the more complicated theorems which still lie before him may be reduced to simpler terms and attacked with success. But it should be noted that the extent of previous reading which is required is really of quite a limited character. If, at every opportunity that presents itself, the student is encouraged to compare geometrical and analytical processes, his appreciation of the significance of the properties he is investigating will be deepened and his resources will be strengthened. Each line of attack is adapted to a special range of circumstance and each has its peculiar needs which may be supplied by the other.

C. V. Durell.

CORRESPONDENCE.

To the Editor of the *Mathematical Gazette*.

Sir,—We have now had approximately ten years' experience of the modern —as opposed to Euclidean—methods of teaching geometry. After such a lapse of time it is only business-like to review the situation.

It is not here my purpose to question the general success of the reform, about which there can be no doubt, but rather to plead for the discussion of a small but vital portion of the subject.

My contention is, reactionary as it may appear, that the absolute elements of geometry should be taught according to *one recognised order*. This, it will be said, would destroy at a stroke the liberty and elasticity of the present methods; but in the preliminary work there can be no such freedom for the individual master; in an average school the mathematical authorities fix the sequence to be followed, and master and boy must conform to it. There is within my knowledge one school where the contrary practice obtained; the beginners in the lower classes encountered no less than three different orders, but the results were not such as to commend the experiment. Apart from this instance, have we not all suffered from and pitied the boy, who, after learning according to one order at a preparatory school, is condemned to follow another later on?

Surely this so-called elasticity, when practised on the beginner, can end in nothing but a truly British muddle, bad alike for the pupil, for his teachers,

and for his examiners. The different methods of approaching the subject have now had a fair trial, and, if geometry is any longer to educate the reasoning faculties, it remains for the Mathematical Association to choose the best method, and to recommend it until a better appears.

Yours faithfully, W. F. B.

ARITHMETIC IN ELEMENTARY SCHOOLS.

DEAR SIR,—Many members of the M.A. must be working in schools in which there are scholars from the elementary schools. The experience of the writer has led him to the conclusion that little or no attempt is made in elementary schools to teach Arithmetic on modern lines. Thus a considerable amount of time has to be taken up in training these boys in methods which should have been placed before them quite at the beginning.

The attention of the teachers in elementary schools should be drawn to the fact that multiplication should be begun with the highest digit and not with the digit of least importance. It is astonishing that the old method is not dead: it is very much alive, especially amongst the female teachers. Pupil teachers who attend secondary schools for a portion of their time where modern methods are in use are sent to be trained to teach Arithmetic in the elementary school to which they are attached. Here the master or mistress persists in the old ways, and when the pupil teachers are called upon to give a lesson in Arithmetic they must not run counter to their instructors, and so the blunders are perpetuated. Enquiry has been made of former pupils, who are at this moment in Training Colleges in London and elsewhere, as to the methods in favour at the schools they attend during their training. "Oh, we are still in the dark up there, we stick to our unit digit." A teacher, who was being helped to prepare for the Certificate Examination, had his attention drawn to the same fact. "Yes," said he, "I have seen it done like that, and I suppose it has its advantages, but when I am giving a lesson I drop into the old method." In cases of this kind what seems to be wanted is a little energetic action on the part of His Majesty's Inspectors. If these gentlemen will administer the necessary amount of shocks things may improve, and if they will steadily set their faces against the use of some abominable text-books much may be done.

Decimals are taken in hand as though the pupils were being initiated into processes absolutely new, and the place values of digits are not properly enforced. As a result, the child who is being stuffed with multiplication sees "one of those d——d dots," as a former Chancellor of the Exchequer is said to have described them, suddenly appear in the product, its position having been fixed by the usual jargon.

In what sort of condition is a boy, who has been so treated, to tackle a simple bit of contracted work? Elaborate directions in carrying out contracted work are not necessary when place values of digits are properly grasped. There is a great saving of time, and logarithms come on the scene without much delay.

"Shop" subtraction is not taught in the place of the ancient method, and Italian Division is ignored, because the teacher has not been accustomed to the method and will not practise it. It is doubtful if any one ever goes back to the cumbrous old method who has once broken away from it. It is sometimes stated that it is more difficult to detect a mistake in division done in the Italian fashion, but familiarity with the method cures this weakness.

The elementary scholar now and again does some wonderful things in division.

It must be remembered that the boys who come from elementary schools to secondary are sometimes the pick of the schools from which they come,

and are generally the holders of County Council and other scholarships. If the early methods of these boys are not up to date there is waste all the way round—waste that can be prevented if those in authority will see to it.—Yours faithfully, T. C. PHILPOTT.

Queen Mary's School, Basingstoke.

THE PILLORY.

DEAR SIR,—I send the following question from the last *Responsions* Arithmetic paper. You may find it worth printing in the "pillory."

The hands of a clock are together at 22 minutes past 4. Is the clock slow or fast, *and how much does it lose or gain in an hour of true time?* The italics are mine.—Yours faithfully, G. P. BLAKE.

Bradfield College.

ANSWER TO QUERY.

[67, p. 330, vol. v.] This theorem is due to Mr. V. Ramaswami Aiyar. It appears in an article contributed to the *Proc. Ed. M. Soc.*, 1896-7.

My proof of the theorem is given in the *Ed. Times Reprint*, Vol. 17, June 1910. It depends on the fact that the common pedal circle of F, F' (the auxiliary circle of the in-conic) cuts the medial or N.P. circle in ω, ω', the *orthopoles* of OF, OF'; *i.e.* the points whose Simson lines (in the medial circle) are parallel to OF, OF'. When ω' coincides with ω, then OF' falls on OF, and hence the well-known theorem: If FF' passes through O, the pedal circle of FF' *touches* the medial circle.

Feuerbach's theorem is, of course, a particular case.

May I be permitted to state what is known at present of this very curious and interesting 'orthopole'?

(The initial N. stands for Professor J. Neuberg; G. for the present writer.)

1. If Ap, Bq, Cr be $\perp$rs on a given line L, then the $\perp$rs from p, q, r on BC, CA, AB respectively are concurrent at a point ω called the orthopole of L. (N.)

2. L being $px+qy+rz=0$ (in barycentric coordinates), ω is given by

$$2\Delta x = q(r-p)ca\cos B - r(p-q)ab\cos C + a^2bc\cos B\cos C. \quad \text{(G.)}$$

3. If L cuts the circle ABC in T, T', then ω is the point of intersection of the Simson lines of T, T'. (N.)

If θ_1, θ_2, θ_3; λ, μ, ν; λ', μ', ν' are the direction angles of TT' and the Simson lines, then for ω,

$$\alpha = 2R\cos\theta\sin\lambda\sin\lambda', \text{ etc.} \quad \text{(G.)}$$

4. For the quadrilateral formed by L and the sides of ABC, the common R.A. of the three diameter circles passes through ω. (N.)

The power of ω for these three circles $= 2d\delta$, where d, δ are $\perp$rs from O and ω on L; also $\delta = 2R\cos\theta\cos\theta_2\cos\theta_3$. (G.)

5. The most remarkable of all the properties of the orthopole is that discovered by M. T. Lemoyne.

Lemoyne's Theorem. The power of ω for the pedal circle of every point F on L is constant. The three diameter circles are pedal circles, so that this common power $= 2d\delta$.

6. Another noteworthy property has been recently published by Prof. Neuberg.

Neuberg's Theorem. If parallel forces $\cos\theta_2\cos\theta_3\sin A$, etc., be applied at the vertices of the pedal triangle of any point F on L, then their centre is a fixed point, the orthopole.

Their resultant $= \Sigma \cos\theta_2 \cos\theta_3 \sin A$
$= \sin A \sin B \sin C.$

This remains true for all lines parallel to L.

7. When L passes through O, ω lies on the medial circle; and since $d=0$, the pedal circle of every point F on L or TOT' passes through ω. (N.)

The Simson line of ω (in the medial circle) is parallel to TOT', and $A'\omega = R\cos\theta_1$.

The co-ords. of ω are: for ABC, $\alpha = p\cos\theta$, etc.;
for the medial triangle $A'B'C'$, $\alpha' = R\cos\theta_2\cos\theta_3$, etc. (G.)

8. If $f_1f_2f_3$ be the pedal triangle of F, then f_2f_3 and $f_1\omega$ meet on $B'C'$. (N.)

There are some other properties, but I dare not ask for more space.

W. GALLATLY.

REVIEWS.

Cambridge Tracts in Mathematics and Mathematical Physics, Nos. 11 and 12. Cambridge: University Press. 1910.

No. 11.

The Fundamental Theorems of the Differential Calculus. By W. H. YOUNG, Sc.D., F.R.S. Pp. 72.

This tract meets a want among English books on the subject. It aims at rigidity of proof and novelty of treatment rather than simplicity of presentation. Naturally this is impossible without the use of the language and concepts of the Theory of Sets of Points. Now the author has solved the difficulty which is involved, not by treating the subject as one branch of the modern Theory of Real Variables, but by studying to make the chain of proofs complete in themselves. This gains aptness and brevity, which are saved from obscurity for the student who has not made an advanced study of the main theory by an appendix of carefully chosen explanations from and references to the existing literature on the subject. The tract deals entirely with functions of real variables, and proceeds from definitions of limits (of all kinds), continuity, differentiation, through a series of connected propositions to the Theorem of the Mean for one independent variable. Then follows the analogous series for two (or more) independent variables. Next there is an instructive section on implicit functions, and the interchange of dependent and independent variables, followed by an important investigation of the reversibility of the order of partial differentiation. Finally, the application of the subject to power series is dealt with, and a full proof of Taylor's Theorem is given. Two features of the tract are the prominence given to the idea of double limits, even in simple differentiation, and the use of Indeterminate Forms to lead up to the Expansion Theorem. The tract well repays study.

No. 12.

Orders of Infinity. The 'Infinitärcalcül' of Paul du Bois-Reymond. By G. H. HARDY, M.A., F.R.S. Pp. 62.

This tract aims at making the ideas of Du Bois-Reymond on Orders of Infinity more widely known and at the same time more up to date. It is practically a formal exposition of the ideas underlying the phrases so often used, 'order of smallness' and 'order of greatness'; and more than this, it develops these ideas into a number of theorems which have wide applications in Pure Mathematics, of which examples are given. First of all a useful notation is introduced and, as in other cases, the notation is a material help in advancing the systematic study of the subject. The foundation of this calculus is a theorem based on scales of infinity: just as series of real numbers can be found of terms in ascending order of magnitude, so series of functions can be arranged in ascending orders of infinity: now the theorem states that "given any ascending scale of increasing functions ϕ_n, we can always find a function f which increases more rapidly than

any function of the scale." The symbolic notation admits of a very precise statement of this and kindred propositions. One extraordinary result which emerges is that the only scales of any practical importance can be made of functions involving a finite combination of the ordinary symbols (viz.: $+$, $-$, $\times$, $\div$, $\sqrt[n]{}$) and the functional symbols $\log(\ldots)$ and $e^{(\ldots)}$—called logarithmico-exponential functions. However functions have turned up in analysis, their laws of increase, if expressible at all, have always been expressible in terms of these functions. There is a detailed investigation of such scales.

In conclusion there is a practical answer to any question as to the bearing of this theory. Thus the tests for Convergency and Divergency of series and integrals obtain thereby not only an elegant form of expression but a new meaning: the theory bears on asymptotic formulae for finite series and integrals, on the theory of prime numbers, on integral functions, and on power series. At the end of the tract are a few numerical illustrations containing some very curious results, also a Bibliography of the whole subject.

The Calculus and its Applications. By R. G. BLAINE, M.E. Pp. 320. London: Archibald Constable & Co. 1909.

This is a practical treatise for beginners and engineers, compiled as the outcome of experience in teaching such students. As such it has several good features: there is no lack of application to problems of all descriptions in which the calculus is a real help; at the same time, many theoretical problems which have no practical bearing, or which are actually untrue in practice because some essential conditions have been overlooked, are studiously avoided. Again, there is an effort to make use of all the simple mathematical theories which lend themselves to practice. For example, the latter sections of the book include methods from the theory of Differential Equations and of Fourier's Series, which are not unduly difficult, and which bear upon all kinds of vibratory motions.

There is an attempt to make the book complete by a useful synopsis of geometrical and trigonometrical results, given from time to time, to allow students, whose mathematical knowledge is very limited, to be free to use formulae derived from algebra, etc., and to meet the needs of those who are working up the subject in private: the quotations indicate what are the most important results to which they must refer back. The book is full of examples, and, where necessary, of hints for their solution.

Zurückführung der Sphärischen Trigonometrie auf die Geometrie des Ebenen Kreisvierecks. By Dr. ERNST ECKHARDT. Pp. 155. (Teubner; Leipzig.)

Spherical Trigonometry has always been a difficult subject to grasp, because it seems at first sight to be simply composed of a mass of more or less complicated formulae: and there appears to be no special reason why the formulae of Napier, Delambre and L'Huilier should be more important than other trigonometrical relations of a spherical triangle.

Dr. Eckhardt has shewn how the subject may be reduced to plane trigonometry by means of a cyclic quadrilateral, and how consequently every type of formula known in the former arises out of obvious relations between the sides and diagonals of a plane quadrilateral. Not only can a conventional spherical triangle (of sides not greater than 180°) be so represented, but the whole group of triangles whose three vertices are given can be correlated to the various cyclic quadrilaterals obtained from given sides a, b, c, d arranged in different orders round the circle. This connects the author's theory with Study's group theory of spherical triangles. It also gives a clear method of solving a spherical triangle when three independent elements are given.

The connection between the spherical and plane figures is effected in this way. If $ABCD$ is a cyclic quadrilateral of sides $AB=a$, $BC=b$, etc., subtending angles y, z, u, x respectively at the circumference, so that $x+y=A$ and so on, then a spherical triangle whose angles are $x+y$, $x+z$, $x+u$ is represented at once. For, calling these angles α, β, γ respectively and the corresponding sides a_1, b_1, c_1, it can be shewn that

$$\cos a_1=\frac{bc-ad}{bc+ad},\quad \frac{\sin a_1}{\sin a}=\frac{\sqrt{abcd}}{\Delta},\quad \tan\frac{a_1}{2}:\tan\frac{b_1}{2}:\tan\frac{c_1}{2}::a:b:c,$$

where Δ is the area of $ABCD$. This leads to an algebraic expression of every magnitude in the spherical triangle in terms of the homogeneous parameters a, b, c, d.

Now there are three different shapes of cyclic quadrilaterals with these given sides, and three different diagonals e, f, g within the circle, where $ef=ac+bd$, $fg=ab+cd$, $ge=ad+bc$. By forming the ratios $\frac{a\pm b}{e}$, $\frac{a+e}{b}$ and similar ones for all combinations of a, b, c, e, f, g we obtain Delambre's analogies. Similarly $\frac{e-a}{e+a}$ is the type which leads to Napier's, and $\frac{e-c+d}{e+c-d}$ to L'Huilier's formulae. It turns out that these well-known analogies are not a complete set, but by taking all the forms of these ratios they can be made complete.

This treatment makes the book very instructive, and throws much light upon the whole subject.

Theorie der Zahlenreihen und der Reihengleichungen. By Prof. Dr. ANDREAS VOIGT. Pp. 133. Leipzig, 1911.

This book is occupied with the algebra of number-rows which naturally arise in the theory of Finite Differences applied to the summation of Arithmetical Series. Starting with an arbitrary number-row we can deduce any number of such rows from the given one by a given law, an example of which is the scheme of Figurate Numbers in which the r^{th} term of the n^{th} row equals the sum of the first r terms of the $n-1^{\text{th}}$ row, the first row being 1, 0, 0, 0, ... Looking upon such numbers as functions in which n is the independent variable, and confining the investigation to two classes of number-rows whose terms are really binomial coefficients, a remarkable theory can grow up with important bearings on the summation of series and in particular $\sum_{r=1}^{n} r^k$, and on the integral solution of indeterminate equations. How this is effected cannot be shortly explained, though the germ of the idea lies in the following:

If N is a positive integer, it has a unique expression in the scale of x;

$$N=a'x^r+b'x^{r-1}+\ldots g'x^2+h'x+k', \text{ say.}$$

This may be transformed into the unique series

$$a\binom{x}{r}+b\binom{x}{r-1}+\ldots+g\binom{x}{2}+h\binom{x}{1}+k,$$

where $\binom{x}{r}\equiv\frac{x!}{r!(x-r)!}$. Thus N has become a linear function of $r+1$ consecutive terms of a number-row, without confining the variable x to any special value.

Perhaps the reason for considering such rows may be most easily illustrated in this way: in the theory of rational numbers the number is rarely looked upon as a single object. Thus to say that a is a multiple of b is really to say that a is a member of the row bx, $(x=0, 1, 2, \ldots)$.

The author has dealt with the subject very fully, and has made the theory of practical value by providing numerical tables to facilitate the actual solution of equations by this method.

Analytical Geometry. By N. C. RIGGS, M.S., Assistant Professor of Mathematics, Carnegie Technical Schools, Pittsburgh, Pa. Pp. 294. (The Macmillan Company. 1910.)

This book has been written with a view to bring out clearly the fundamental principles and methods of the subject, and to make it a natural introduction to more advanced work. If analytical geometry means simply the analytical discussion of the geometry of a single curve considered as the locus of a moving point depending on one parameter x or t, then the book has served its purpose well. For there is a splendid introduction explaining the use of linear measurements to represent number, and the ideas involved in projection, coordinates and a geometrical locus. This is followed by a practical treatment of the simpler equations which represent curves, algebraic and transcendental, all with a view to its application to the calculus. Prominence is given to the parametric representation of the cartesian coordinates of a point on a curve, which is always instructive and forms a good foundation for more advanced work. Then the

book concludes with an introduction to the calculus and a short account of three dimensional coordinate geometry.

If, however, one takes a broader view of the meaning of analytical geometry the book fails signally in being an introduction to more advanced work. The subject always seems to begin with the meaning of the equation $\alpha - \lambda\beta = 0$, where α and β are short for linear functions of x and y. There are plenty of simple uses to which this idea leads, so that a book need not omit it because of its difficulty. At the same time it helps to form the idea that points are not the only geometrical elements which can be specified by coordinates. The same might be said of the analytical representation of coaxial circles as $S - \lambda S' = 0$. Now this book omits all mention of such systems of equations. It would therefore not be so suggestive an introduction to a beginner with a liking for geometry as some of the older books on the subject.

H. W. TURNBULL.

The Teaching of Arithmetic. By D. E. SMITH, LL.D., Professor of Mathematics in Teachers College, Columbia University. Pp. 120. 75 cents. 1910. (Teachers College, Columbia University.)

Arithmetical Abilities and some Factors determining them. By C. W. STONE, Ph.D. Pp. 101. $1.00. 1909. (Teachers College, Columbia University.)

If modern American text-books in arithmetic offer any criterion of the teaching of the subject in the States, it would appear that America lags behind this country, and indeed most European countries. For the most part they are characterised by a drab monotony of treatment, by a lack of vitality, and by an entire absence of any correlation with other branches of a mathematical education. Professor Smith's book affords ample confirmation of these views, and yet he leads among the reformers in his country. In his general remarks upon the teaching of the subject he is sound, and what he has to say is worthy of the attention of all who are grappling with this subject. He sees the defects in American teaching, he diagnoses the weakness, but, in our opinion, he fails to find proper remedies. A complete treatise would be needed to treat the question adequately, and in the short space of a review it is not possible to do more than to indicate briefly what we consider to be two or three defects in Prof. Smith's treament of the subject. In the first place, we find no suggestion of anything in the way of practical work, a prominent feature in modern arithmetical teaching in this country. Tables of measurement, the fundamental operations, the use of decimals, elementary mensuration, and kindred matters, seem to be dealt with entirely in an abstract fashion from the blackboard. In the second place, there seems to be no suggestion of the desirability, by means of the practical work and otherwise, of linking up the arithmetic with geometry, algebra, mensuration, or physics. Thirdly, we believe that the monotonous character of the examples must fail to provide that stimulus which is necessary for intelligent work in the subject. Prof. Smith asks for the rejection of the academic type of example which America has inherited from Italy *via* England, and suggests more practical examples. If they are to be of the types which he gives as specimens, we doubt whether there will be much improvement, for his "practical" examples seem to be confined largely to trade and agriculture. We cannot imagine any English boy being stimulated by such examples as: "Silver is sold by the troy ounce. This is what per cent. of the avoirdupois ounce (7000 gr. = 16 av. oz., 5760 gr. = 12 troy oz.)?" Or "When Iowa's annual product amounted to 305,800,000 bu., this was how many times the 440,000 bu. produced by Maine?" Perhaps the American boy thrives better than the English on such fare. It seems as though the American teacher has yet to learn that the phenomena of our daily existence, our simplest acts and experiences, present a rich field for those seeking for quantitative problems which will enliven the arithmetic lesson, quicken the intelligence of the pupil, and lead him to feel that his school arithmetic is intimately associated with the world in which he lives and acts.

Dr. Stone's book affords evidence that American educationists are addressing themselves seriously to the scientific study of pedagogy. It is an account of a series of experiments undertaken in a number of American primary schools with a view to an examination of the factors determining arithmetical ability. The method of conducting the investigation is interesting, and similar work might

well be undertaken in our own country. The results of the investigation do not, however, reveal much that is new or startling; they merely confirm what is common knowledge of most of those who teach arithmetic. The following quotation will bear this out: "Doubtless the most helpful generalization possible from this study is that there is no *one* factor that produces abilities, there is no single 'summum bonum' in teaching arithmetic. The course of study may be the most important single factor, but it does not produce abilities unless taught. The other essential conditions for successful teaching are children and teachers of usual abilities, a reasonable time allotment, intelligent supervision, and adequate measuring of results by tests." P. A.

Elementary Algebra Exercises. By A. E. Layng. (Blackie.) Pp. 165.

These are the exercises only of Mr. Layng's work on Elementary Algebra. They form a convenient class-book for the use of members of that ever-increasing body of schoolmasters and tutors who prefer to be their own text-book, either for economic or pedagogic reasons. The usual subject matter of algebra, up to and including the exponential theorem, is represented by a good choice of more or less original exercises, and at the end of the book are about 350 miscellaneous examples and a log. table.

A First Course of Practical Mathematics. By B. A. Tomes. (Blackie.) Pp. vii + 188.

This book is designed to suit the requirements of the highest standards in the Primary School, and of the preliminary classes in mathematics in Evening Continuation and Technical Schools. The author has profited by a wide experience with such classes, with the result that his book seems an ideal one for its purpose. It has evidently been carefully arranged, and special prominence has been given to the correlation between the different branches of mathematics that may be connected with the fundamental idea under consideration. The final product is a good progressive course in arithmetic, algebra, geometry, and trigonometry, and can be recommended to those in charge of technical students beginning mathematics.

Four-figure Logarithms: on a New Graphic System dispensing with Interpolations. By R. C. Farmer and M. M. Farmer. (Longmans.) 1910.

The title of this little book raised great hopes, which, however, were not realised on examination of the contents. The idea seems a good one. In it, the multiples of 10 from 1000 to 10000 and their logs. are set side by side in columns, with a vertical scale between them. The effect at the beginning of the scale is passable, but towards the end the sub-divisions become microscopic, and are hardly suitable for regular use by young people. The method of arrangement is a good graphic illustration of interpolation, and might be helpful as such. A table of some useful constants is included.

A Course of Practical Physics. By E. P. Harrison. (Longmans.) 1910. Pp. x + 194.

This excellent book, by the Professor of Physics in the Presidency College of Calcutta, is practically the third and fourth year course in Physics at the University, and easily covers the syllabus in this subject for the B.Sc. It can be recommended for the corresponding London Examination, though some tables of constants, and one or two instructions in the text, would have to be modified to suit the different conditions of laboratory temperature, etc. The mathematical theory of the method accompanies the instructions for carrying out each experiment, and this by itself would constitute a useful text-book. There is a misprint on p. 176.

The Senior Arithmetic. Parts I. and II. complete. Edited by J. L. Martin. (Harrap.) 1909. Pp. iv + 128 and iv + 208.

This is the Senior course of a "New Concrete Practical Arithmetic," and all the adjectives are appropriate. The usual dreary tale of monotonous examples of the ordinary arithmetic class-book is replaced by sets of bright every-day problems, and even where types have to be considered, the exercises are made interesting. The captious critic could find fault with one or two methods of

procedure and arrangement, but the book is obviously in the right direction; and its low price and general appearance will help to further its chances of success. There is a misprint on p. 64. G. H. M.

Précis de Mécanique Rationelle. Par P. APPELL et S. DAUTHEVILLE. (Paris: Gauthier Villars, 1910.) Pp. 729. Price 24 fr.

The present volume, as its name implies, is an abridgment or précis of the larger treatise by M. Appell in three volumes.

Two topics on which the book throws some light are of growing importance in relation to advanced school or elementary undergraduate work on mechanics. The possibility of simplifying the traditional course in mechanics by the use of vectorial methods and notations—alas that notations must be plural!—is one; the possibility of paving the way for the mathematics of electricity is the other.

Like the larger treatise, the present book uses vectorial notions—but not vectorial algebra, while reference is constantly made to electrical and magnetic analogies, and at page 294, for instance, some particular cases of the motion of a charged particle in an electric and magnetic field are worked out as examples of the integration of equations of motion.

Does any English book on the subject of rational mechanics mention the name of Faraday? (p. 567.)

The book is divided into three parts, dealing with vectors and preliminary theorems in kinematics, statics, and dynamics—the latter comprising particle and rigid dynamics, potential, hydrostatics and hydrodynamics.

A set of 173 examples set at various examinations concludes the work.

Les Sommes de $p^{\text{ièmes}}$ puissances distinctes égales à une $p^{\text{ième}}$ puissance. Par E. BARBETTE. Liege. 1910. E. Gnusé. 4to. Pp. 154 and 12. Price 12 fr. 50.

As is well known, Dr. Wolfskehl bequeathed a prize of 100,000 marks for the discoverer before 2007 A.D. of a proof of Fermat's "Last Theorem," that x^n+y^n and z^n are inconsistent forms of a positive integer if n is an odd prime. The modern interest thereby conferred upon the time-worn theorem has called forward a considerable number of knights of the post: ready and willing to prove anything on reasonable terms.

It is therefore due to Dr. Barbette to state that he does *not* prove the 'Last Theorem' and that his researches have extended over many years prior to the foundation of the Wolfskehl prize.

Dr. Barbette deals with the problem of factorization and the utilization of triangular numbers $\left(\text{numbers of the form } \frac{n.n+1}{2}\right)$ for this purpose.

He cites another of the puzzles so plentiful in Fermat's works. In a letter to Mersenne, Fermat remarks [*Œuvres*, Vol. 2, p. 256], "You ask me whether 100, 895, 598, 169, [say u] is prime or not. To which I reply that it is composite: the product of 898423 and 112303." How were these factors obtained by Fermat? M. Barbette obtains them by noting that $4u$ is a triangular number and again by noting that $32u+1$ is a square.

The remainder of the book deals chiefly with the representation of a p^{th} power as the sum of p^{th} powers. Dr. Barbette shows for example how to obtain sets of consecutive cubes whose sum is a cube, and, by systematic but somewhat tentative methods obtains a large number of identities such as

$$11^3+15^3+27^3=29^3,$$

$$4^4+6^4+8^4+9^4+14^4=15^4,$$

$$4^5+5^5+6^5+7^5+9^5+11^5=12^5.$$

The work terminates with a table of the first 5000 triangular numbers and with a list of the primes less than 10,000.

Though perhaps it may be doubtful whether the author has thrown very much additional light on the problem of dealing with exceedingly large numbers, his book represents a vast amount of patient research, and will be viewed with respect and sympathy by all who have in their small way spent pleasant hours in similar occupations.

Civil Service Test Papers in Mathematics with Answers. By A. F. VAN DER HEYDEN. Pp. 118. Middlesborough: Appleyard & Sons. 1910. Price 3s. 6d.

A collection of 315 questions, of the type rendered familiar by Mathematics I. in Army examinations, and the examinations for second division clerkships. Of the questions 224 are original, and the remainder are from recent examinations. Some of the questions are very neat, particularly one on Peaucellier's linkage (p. 34). At p. 43 it is not quite clear how much steel there is in the structure, nor what it cost. The arrangement and general style of the questions should render the book well suited for use by candidates preparing for examinations of the kind mentioned.

Solid Geometry. By C. GODFREY and A. W. SIDDONS. Pp. 109. Cambridge: University Press. 1911. Price 1s. 6d.

If we compare the abandonment of Euclid to the escape out of Egypt, most teachers would acknowledge, and the authors would probably be among the first to agree with them, that we are still wandering in the wilderness. The decision of many questions must await the result of patient investigation, in which due account will be taken of the natural development of the growing intellect, of the practical conditions and limitations of the schoolroom, and of the modifications of scientific opinion. For instance, since a beginner's first acquaintance with geometrical notions must necessarily be derived from solid objects it would seem that the Italian school of Fusionists,* who would treat plane and solid geometry together from the start, must be seriously reckoned with. At present, however, a growing opinion that solid geometry calls for increased attention is as much as can be counted upon. Hence we find proposals to demand merely an acquaintance with the forms of the simpler solids, or to follow a course of plane geometry with a modified version of the eleventh book, or with a scheme based on the practical applications of descriptive (Mongian) geometry, or with a sketch of modern projective methods. Under these circumstances the authors have probably been judicious in writing themselves down eclectic, giving brief sketches of several topics, and leaving considerable latitude to the teacher, who will find well chosen and well ordered materials for whichever of these alternatives he may prefer.

A feature, which those who know the earlier works by the authors will expect, is a large number of well-chosen, easy and well-arranged examples. One matter seems to invite discussion. In notes at several places in the text the authors recommend dealing with *numerical* examples by "rough sketch" rather than "accurate drawing," and justify themselves in the preface thus:

"If the aim is educational rather than technical, very accurate drawing is hardly necessary; whatever educational benefit is to be gained from drawing accurately has presumably been gained during the earlier study of plane geometry. But it must be remembered that in some cases a fair degree of accuracy is needed to reveal the essentials, *e.g.* there are often cases of concurrency or collinearity the failure of which would wreck the figure."

Now, it may be at once admitted that in matters educational there are so many considerations to be had in mind that there may be a great deal of sound common-sense in the assertion, and in the denial, of a bare principle. Moreover, anyone acquainted with the ways of professional draughtsmen will notice that some of them show more skill in concealing mistakes than in avoiding them. But it does seem that an exactly opposite argument of some cogency might be framed as follows:

If the pupil has gained any educational benefit from his early study of the use of instruments he will realise the value of a neatly drawn diagram, and will have very little trouble in constructing one. Moreover, in descriptive geometry it is so generally possible to check the work by drawing one checking line, that accurate figures are of peculiar importance. Professor Klein has observed that a time may come when we shall put correct drawing on a level with correct computation.

The authors have produced in an unpretending form some excellent material, and have in many of the exercises combined very happily a demand for space intuition, and for some power of Euclidean reasoning.

*The treatise of Professor Lazzeri and Bassani has recently been translated into German by Herr Treutlein.

Diophantus of Alexandria, a Study in the History of Greek Algebra. By Sir Thomas L. Heath, K.C.B. Second edition, pp. vi+387. Cambridge University Press. 1910. Price 12s. 6d. net.

The first edition of this work was published in 1885, and has for many years been unprocurable. Since that date the materials available for the student of Diophantus have largely increased. In 1890 Wertheim gave an amplified version in German with brief commentary. In 1893 Tannery edited a Greek text, with a close Latin translation on the opposite page, a very convenient form for those of us whose Greek never got beyond the Little-Go standard.

Meanwhile the national edition of the works of Fermat, the great master of this region of algebra, has been well nigh completed. These materials have been fully utilised by Sir Thomas Heath in this new edition, and in particular special attention has been devoted to Fermat.

It is well known that several of Fermat's more celebrated results were recorded by him in the form of notes on the margin of Bachet's commentary on Diophantus, and that he expressly assigns the "narrowness of the margin" as a reason for not putting on record his "wonderful proof" of the impossibility of representing any power but a square as the sum of like powers, and makes a similar statement about the "wonderful and very general theorem" that every number is the sum of three triangulars, four squares, five pentagons . . . at most.

The celebrated saying that God made the integers conveys the idea which renders the study of Fermat so curiously fascinating. If we look upon the series of natural numbers, traces of structure are manifest in the distribution of primes, in the existence of representations as sums of two, three and four squares, in sequences of non-prime numbers. Now a number has an existence quite apart from its representation in the scale of 10, or as a quadratic form of a particular type. Had Fermat discovered some completely new aspect in which to arrange and exhibit integers of various types, or had he merely a peculiar talent of the kind possessed also by Euler, in devising algebraical transformations for the solution of Diophantine problems such as (p. 241) to show that $x=5, y=3$, are the only integers which satisfy $x^2+2=y^3$, problems which the Philistine would call curious if he wished to be polite?

Sir Thomas Heath gives at p. 145 a cautious and guarded approval of Wertheim's doubt (Wertheim, p. 52) as to whether Fermat really possessed a proof of his "last theorem" that x^n is never of the form y^n+z^n.

The argument on the other side was powerfully stated by H. J. S. Smith, Report on Theory of Numbers, p. 149, in a form to which an abridged quotation does scant justice.

". . . We have the assertion of Fermat himself that he was in possession of the demonstrations: and although when we consider the imperfect state of analysis in his time, it is surprising that he should have succeeded in creating methods which subsequent mathematicians have failed to rediscover, yet there is no ground for the suspicion that he was guilty of an untruth or mistook an apparent for a real proof . . . It would be inexplicable, if his conclusions reposed on induction only, that he should never have adopted an erroneous generalisation, and yet with the exception of the 'last theorem' every proposition of Fermat's has been verified by the labours of his successors. There is indeed one exception to this statement, but it is an exception which proves the rule. Fermat, in a letter to Sir Kenelm Digby, stated that $2^{2^n}+1$ is always prime, acknowledging that he had not succeeded in obtaining a demonstration. Euler subsequently showed that $2^{2^5}+1$ is divisible by 641. The error, if it is an error, is a fortunate one for Fermat: it exemplifies his candour and veracity, and shows that he did not mistake inductive probability for rigorous demonstration. 'Mais je vous advoue tout net,' are his words, '. . . que je n'ay peu encore démonstrer l'exclusion de tous diviseurs en cette belle proposition. Car bien que je reduise l'exclusion à la pluspart des nombres, et que j'aye même de raisons probable pour le reste je n'aye peu encore démontrer necessairement le vérité de cette proposition.'"

It is tempting to endeavour to follow the author to other topics. But a review which attempted to indicate all the interesting questions discussed in text, notes, and supplement would occupy a whole *Gazette*. It must suffice to say that English students are now provided with a translation of Diophantus, and with the essence of everything worth preserving which has been written on the subject since. C. S. J.

The Thirteen Books of Euclid's Elements. Translated from the Text of Heiberg, with Introduction and Commentary, by SIR T. L. HEATH, K.C.B., D.Sc., Sometime Fellow of Trinity College, Cambridge. Vol. I. Introduction and Books i. & ii. Pp. x+424. Vol. II. Bks. iii.-ix.; pp. 436. Vol. III. Books x.-xiii., and Appendix; pp. 554. (Cambridge: University Press). £2 2s. net.

It would be interesting to write an essay on the manner in which officials of the State have employed their leisure hours. We might well enquire for instance how it was that the poetical genius of an Austin Dobson survived an exposure of five and forty years to the uninspiring atmosphere of the Board of Trade. The old days of "office hours from ten to four" have been relegated to the limbo of the past, but even with this restriction upon their time we can point to many an instance, in the Indian and Home Civil Services, of men whose intellectual activities have not been allowed to rust, and who in the limited periods of leisure at their disposal have done work of the highest quality in their respective spheres of interest. Among those who have thus cast an additional lustre upon the great Services of which they are distinguished members, may be numbered Sir T. L. Heath, who has certainly not misused the moments he has snatched from the heavy pressure of his duties at the Treasury. In the year that his brother was Second Wrangler and Second Smith's Prizeman, he took a First Class in the Classical Tripos; the next year he was Twelfth Wrangler, and the year after that he was placed with two distinctions in the Second Part of the Classical Tripos. Small wonder that he came out at the head of the Home Civil Service competition of the next year, and that his subsequent career enables us to place him among those of whom Dr. John Dee has said:—"Happy are they that can perceive and so obey the pleasant call of the mighty Lady Opportunitie." From the moment the claims of the examination room were relaxed he devoted himself to, or rather, resumed his researches into the history of Greek Mathematics. The year after his appointment to the Treasury he published his *Diophantus*, of which a new edition has just appeared. A further twelve years' labour saw his *Apollonius* and *Archimedes*. And now after a like period we have before us a fine achievement represented by the three stately volumes before us, which remove from the English mathematical world a reproach to which it has long been exposed. We are no longer ashamed in this respect to speak with our friends the enemy in the gate. To Sir T. L. Heath we owe a debt that cannot readily be repaid.

"Mathematics," said Cayley, "have steadily advanced since the days of the Greek geometers. Nothing is lost or wasted; the achievements of Euclid, Archimedes, and Apollonius are as admirable now as they were in their own days." And this is true to a very great extent of a host of other figures of which, as the Platonic dialogue hath it, we see the shadows as they flit across the stage. The author cannot be too cordially congratulated on the admirable skill with which he has wielded the enormous mass of material with which he has had to deal. He has impressed into his service, one would think, almost every thing of any interest that has ever been written or said of the great Alexandrine classic, by men in any tongue or in any clime. And yet all is handled with such consummate dexterity that the amorphous mass assumes before our eyes the proportions of an exquisitely modelled statue. Bold speculations, rash conjectures, dim half-revealed gropings, are dealt with both tenderly and with a just discrimination of the part they played in the general development. And he has a shrewd eye, as de Morgan might say, for defects that are revealed by the light of excellencies. Both in his Introduction and in the running Commentary the author shows that he has, so to speak, at his fingers ends the results of the researches of the moderns from Cantor to Björnbo. But there is also ample internal evidence of first-hand knowledge of the material with which he deals. And perhaps we can award these volumes no higher praise than to say that this sheer weight of erudition in no way hinders the flow of this absorbing history of one of the world's great classics. Even in the Chapter on Translations and Editions which to some readers may seem likely to be the dullest chapter in the book—and to the true bibliophile how delightful it is!—will be found many a light touch to take away the sense of heaviness from its perusal. We hear how Martianus Capella speaks of the effect of the enunciation of the proposition "how to construct an equilateral triangle upon a given finite straight line" upon a company of philosophers, who recognising the first proposition of the Elements, straightway break out into encomiums on Euclid. The

same Martianus gives as the definition of a point:—punctum vero est cuius pars *nihil*. It will be news to some readers that the first mention of Euclid in Latin is by Cicero (in the *de Oratore*) and that the introduction into England of Euclid in at least fragments of a translation must have been as far back as 929-940 A.D. Zamberti who brought out the first translation of the whole of the Elements from the Greek scoffs at previous rivals, amongst others at "that barbarous translator" who filled a volume purporting to be Euclid's, "with extraordinary scarecrows, nightmares and phantasies." In his turn Zamberti receives the lash at the hands of Paciuolo, who expresses the pious wish that "others too would seek to acquire knowledge instead of merely showing off, or that they would not try to make a market of the things of which they were ignorant, as it were (selling) smoke." We notice by the way that on p. 111, it is stated that 1714 was the date of Whiston's abridged version of Tacquet's edition of the Elements. The first edition of Tacquet published by Whiston was in 1702, the year before the appearance of David Gregory's Greek-Latin Oxford edition. There were many editions of Tacquet-Whiston, but that of 1702 seems to have escaped the author's notice. As an instance of the care that has to be taken in the matter of dates we may mention a case apropos of Mr. Whittaker's new book *On the Theory of the Aether*. He gives the date of Hooke's *Micrographia* as 1667, with a note to the effect that the imprimatur of Lord Brouncker, then President of the Royal Society, is dated 1664. The date 1664 is followed by Barlow and the *Encyclopaedia*, while the *National Dictionary of Biography* gives it as 1665. The mystery surrounding the figures of the propositions in Ratdolt's edition of 1482, which was not only the first printed edition in Latin of the Elements, seems (p. 97) to be still unravelled. It is interesting to note apropos of Billingsley's "Elements of Geometrie of the most auncient Philosopher EUCLIDE of Megara" (1470) that in Bk. xi. in addition to Euclid's figures, pieces of paper (triangular, rectangular, etc.), are "pasted at the edges on to the pages of the book so that the pieces can be turned up and made to show the real form of the solid to be represented." We have often wondered why writers of editions of Euclid have not adapted in some form or other this simple device. It is used for another purpose by Sir William Ramsay in his little volume on *Modern Chemistry*. We would very much like to see a reprint of the "fruitful Preface" by Dr. John Dee, "where are disclosed" in his inimitable way "certain new Secrets Mathematicall and Mechanicall, until these our daies, greatly missed." As far as we are aware it has not been reprinted since the days of gallant Captain Rudd, who inserted it in his *Elements* in 1651.

In the pages dealing with the Euclidean tradition the author points out how the confusion arose between Euclid the geometer and Euclid of Megara, a misunderstanding from which but one writer seems to have been free until towards the end of the sixteenth century.

The Second Chapter will probably be found to contain much that is quite new to many of our readers, for it is devoted to the Euclidean treatises other than the Elements. There is the Pseudaria, constructed, as Proclus says, "for cathartic and exercise." This we are never likely to see, unless in some miraculous way it shall be found to be preserved beneath the lava of Herculaneum. Both this treatise and the data are concerned with elementary geometry, but the latter volume also introduces the student to higher analysis—"may in fact be regarded as elementary exercises in analysis." The book on Divisions (of Figures) "is lost in Greek but has been rediscovered in the Arabic." Then there are the three lost books of the Porisms, which have exercised the ingenuity of so many, from Pappus (though the information derived from that source is so fragmentary and imperfect as to be "unintelligible" to the acuteness of a de Morgan), and Proclus, down to Simpson and Chasles, and in later days to Heiberg, Loria and Zeuthen. Two books on Surface-loci, four on Conics, the eighteen propositions of spheric geometry which form the Phaenomena, the Optics, and the Elements of Music complete the list.

Chap. iii., deals with the Greek Commentators on the Elements other than Proclus, who has Chap. iv. all to himself. The discussion of the Text of the Elements takes sixteen pages, and ten are given to the Scholia. Chap. vii. on Euclid in Arabia is a mine of information for those not conversant with German and other sources. Chap. ix., a careful study of the technical terms. The ground covered is indicated by its sections:—the nature of elements, elements anterior to

Euclid's, definitions, postulates, and axioms, theorems and problems, the formal divisions of a proposition, real and nominal definitions. This completes the introduction, which is in itself a masterly performance, and we hope that some day the author will see his way to publishing it in a separate form.

Then follows the author's translation from the Greek text, with a critical commentary to the value and interest of which we can hardly do justice. Enough to say that it is the work of a savant, using the word in its widest sense. Forty pages of small print are given to the Definitions, twenty-five to the Postulates, and twenty to the Axioms or *Common Notions*. The nineteen pages devoted to the famous Postulate Five give the author an opportunity of giving his due to Saccheri as "the first to contemplate the possibility of hypotheses other than that of Euclid, and to work out the consequences of those hypotheses. He was therefore a true precursor of Legendre and of Lobachewsky, as Beltrami called him (1889), and it might be added, of Riemann also. For as Veronese observes.... Saccheri obtained a glimpse of the theory of parallels in all its generality, while Legendre, Lobachewsky and G. Bolyai excluded *à priori*, without knowing it, the 'hypothesis of the obtuse angle' or the Riemann hypothesis. Saccheri, however, was the victim of the preconceived notion of his time that the sole possible geometry was the Euclidean, and he presents the curious spectacle of a man laboriously erecting a structure upon new foundations for the very purpose of demolishing it afterwards; he sought for contradictions in the heart of the systems he constructed, in order to prove thereby the falsity of his hypotheses." Killing's discussion of the Principle of Continuity with reference to Dedekind's postulate and its applications receives full attention, and the summary of Hilbert's system of congruence-postulates is given with the hint that it is of peculiar interest to those engaged in the teaching of geometry. Passing over the first four books, the commentary on which teems with suggestive material not only for every teacher but for all devotees of the Queen of the Sciences, we come to Book v. "Considered as a complete conquest over a great and acknowledged difficulty of principle, this book of Euclid well deserves the immortality of which its existence, at the present moment, is the guarantee; nay, had the speculations of the mathematician been wholly confined to geometrical magnitude, it might be a question whether any other notions would be necessary." In his illuminating notes on the Definitions of this book, the author might have appended to de Morgan's illustration a reference to the convenient notation due in the main to Prof. Love, and given in Prof. Hill's *Contents of the Fifth and Sixth Books*, pp. 13-15.

We imagine that to many teachers the portions of these volumes dealing with the arithmetical Books vii. to ix., and Euclid's elaboration of irrationals in Book x., will afford considerable astonishment if it has not been their fate to meet with them before in some form or another. In his article in the *Penny Cyclopaedia* on Irrational Quantity, de Morgan says of the tenth book:—though Euclid was not acquainted with any direct algebraical process, yet he carried the distinction of incommensurable quantities to the length of a complete sub-division of all the possible cases which can be contained in the formula $\sqrt{\sqrt{a} \pm \sqrt{b}}$. We are induced to give an account of his Tenth Book, because there does not to our knowledge exist any such thing in a form accessible to the student. Indeed we do not know where to find a description of its details in any form whatever.... The preceding enumeration points to one of the most remarkable pages in the history of geometry. The question immediately arises "Had Euclid any substitute for Algebra?" There will not be many lovers of geometry to whom the study of the Thirteenth Book will be devoid of interest, and if in the audience to which the *Gazette* appeals there are any members of that esoteric body known as the "Royal Arch," we would suggest that it is almost their bounden duty to see what is here to be said about the five regular solids, the famous Platonic figures, as they are inaccurately termed. They will be interested to learn from these pages that dodecahedra have been found, of bronze or other material, which may belong to centuries earlier than the days of Pythagoras.

And here we must pay our tribute to the excellence of the diagrams throughout the volumes, and notably in the Thirteenth Book. The Appendix deals with the contents of the so-called Book xiv., by Hypsicles, and there is a note on the so-called Book xv. The frontispiece is an excellent facsimile of a page of the

Bodleian MS. of the Elements, written in the year 888. The reader of vols. i. and ii., must not forget to keep his eye on the six pages of addenda and corrigenda at the end of vol. iii. Although continued pressure of space has prevented an earlier publication of this imperfect notice it cannot be too late to assure the author of the full appreciation which all students of mathematical history will have for the successful issue of his labour of love. And when a labour of love happens also to be a labour of prolonged and toilsome research, the appreciation of the work accomplished is enormously enhanced by the feeling of gratitude aroused in the minds of serious students. Those who have succeeded in reaching even the frontier of the vast and distant region which Sir T. L. Heath has traversed and from which he brings back to us such inspiring ideas and such an imposing array of material—the spoils of long years of untiring and unremitting effort, will realise what drudgery he has spared to his successors.

We have detected no misprint more serious than "geometrie non-euclidee" (p. 219). The indexes are very full. An English index is given to the first two volumes, with one for Greek forms and words, as well as a final general Index to the book as a whole In a second edition, to the Index heading *Henrici and Treutlein* may be added, III. 489.

Primer of Statistics. By W. P. and E. M. Elderton. Pp. 86. 1909. (Black.)

To familiarise "educated persons with the most recent developments of the new school of statistics" is the aim of this valuable little primer. Mrs. Elderton is a Galton Research Scholar in National Eugenics, Mr. Elderton is a Fellow of the Institute of Actuaries, and the primer was issued to the world with the blessing of Sir Francis Galton. The "educated person" will not be led astray if he begins his study of statistics under such guidance. The scope of this handy and useful manual will be gauged from the titles of the chapters: variates and medians; quartiles and medians; frequency distributions; mode—standard deviation—coefficient of variation; correlation of probable errors. The treatment is as simple and clear as it can be made, and while the teacher should be familiar with the subject matter, both as "educated person" and in his professional capacity, the boy will find the study of most of the contents a very agreeable holiday task. Few boys will not revel in Chapter iv., which studies amongst other things the two problems: What was Tunnicliffe's most likely score in 1906? Was he a consistent batsman? Warner's scores are utilised in the same way to exhibit the meaning of the "sort of" average measure of deviation, to which is given the name of standard deviation.

Homogeneous Coordinates. By W. P. Milne. Pp. xii + 164. 5s. net. (Arnold.) 1910.

"Processes and methods rather than results" is the keynote to the little monograph published by Dr. Milne. The Honours candidate at the Universities and the student who hopes to win a mathematical scholarship will welcome its appearance, for it fills one of the proverbial gaps. Of the 164 pp. nearly fifty are given up to examples selected from Tripos and Scholarships papers and other sources. So that we are left with about 120 pages of text on homogeneous coordinates in general, with a due regard for areals, trilinears and tangential coordinates—just about double the number of (somewhat larger) pages devoted to the same topics in the latest of English serious works on Analytical Geometry, that of Dr. Askwith. The chapter on parametric representation will at once attract the attention of the teacher as one long needed. The statement (in black type) without proof of the various geometrical theorems to be handled is accompanied by references to a limited number of text-books, *e.g.* Salmon, Filon, Russell, Smith, Edwards, and Miss Scott, which every teacher of those who are specialising in mathematics is sure to have in his possession or at his disposal in a properly equipped school library. Thus, the existence of the line at infinity is admitted on the authority of Miss Scott, and its equation is found in areals. It might have been well to give the proof that every point at infinity is on that line, and that every point on the line is at infinity, or at any rate to have suggested after section 12 that the student should consider these statements. We are somewhat dubious as to the manner in which the author has handled the material in the chapter on the circular points at infinity—*i.e.* as to the effect his presentation will have on

the mind of the boy who has been trained on modern lines, and acquired a liking for realities and to see the how and the why of things. Here perhaps wider experience will dictate another method of approach, and for that we must "wait and see." In any case the scholarship candidate cannot do without Mr. Milne's carefully compiled volume. The diagrams are good, the print large, and the lines well spaced, so that as far as physical conditions are concerned the book will be consulted with pleasure.

THE LIBRARY.

Wanted by purchase or exchange:

1 or 2 copies of *Gazette* No. 2 (very important).
1 copy „ No. 3.
2 or 3 copies of Annual Report No. 11 (very important).
1 or 2 „ „ Nos. 10, 12 (very important).
1 copy „ Nos. 1, 2.

ERRATA AND ADDENDA.

P. 350. The second fraction in (D) should be $\frac{3}{10}$.

P. 351. 8th line from the bottom: *for* "greatest term l," *read* "greatest term l^n."

P. 352. 3rd paragraph from the bottom:
for "and the square roots of the terms of the given series"
read "of the terms of the given series and afterwards squaring them."

P. 354. About the middle:
for "5, 4, 3, 2, 1, 1, −1, −2, ..."
read "5, 4, 3, 2, 1, 0, −1, −2, ..."

P. 379. Line 17:
for "$r-2r^{\frac{1}{2}}a^{\frac{1}{2}}-a$, $r^{\frac{3}{2}}-3ra^{\frac{1}{2}}-3r^{\frac{1}{2}}a-a^{\frac{3}{2}}$,"
read "$r-2r^{\frac{1}{2}}a^{\frac{1}{2}}+a$, $r^{\frac{3}{2}}-3ra^{\frac{1}{2}}+3r^{\frac{1}{2}}a-a^{\frac{3}{2}}$,".

P. 380. Line 2: *for* "number" *read* "numbers," and *for* "indicates" *read* "indicate."
Line 3: *for* "first" *read* "second."
Line 5: *for* "index" *read* "indices."

P. 381. Foot-note, line 3: *for* 135 *read* 185.

P. 382. Line 2: *for* "numbers" *read* "number."

P. 383. Towards end: *for* $\frac{2002}{2002}$ *read* $\frac{2002}{2001}$.

P. 384. In the series *for* $\frac{\pi}{4}$ the third numerator should read 1×3 and the fourth $1\times3\times5$.

BOOKS, ETC., RECEIVED.

The Mathematics Teacher. March, 1911. Published Quarterly by the Association of Teachers of Mathematics for the Middle States and Maryland. Pp. 100-1641. $1 per ann. Edited by W. H. METZLER. (41 North Queen St., Lancaster, Pa.)

La Escuela Mexicana. March 10th, 1911. Vol. VII. No. 34. Edited by Prof. BRUNO MARTINEZ.

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